

# CHOW GROUP OF 0-CYCLES WITH MODULUS AND HIGHER DIMENSIONAL CLASS FIELD THEORY

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ABSTRACT. One of the main results of this paper is a proof of the rank one case of an existence conjecture on lisse  $\overline{\mathbb{Q}}_\ell$ -sheaves on a smooth variety  $U$  over a finite field due to Deligne and Drinfeld. The problem is translated into the language of higher dimensional class field theory over finite fields, which describes the abelian fundamental group of  $U$  by Chow groups of zero cycles with moduli. A key ingredient is the construction of a cycle theoretic avatar of refined Artin conductor in ramification theory originally studied by Kazuya Kato.

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## INTRODUCTION

One of the main results of this paper is a proof of the rank one case of an existence conjecture on lisse  $\overline{\mathbb{Q}}_\ell$ -sheaves on a smooth variety  $U$  over a finite field suggested by Deligne [EK], which was motivated by work of Drinfeld [Dr], see also [D]. The conjecture says that a compatible system of lisse  $\overline{\mathbb{Q}}_\ell$ -sheaves on the subcurves of  $U$ , satisfying a certain boundedness condition for ramification at infinity, should arise from a lisse sheaf on  $U$ . A precise formulation is given in Question IV below.

**Class field theory.** First we state our main result in the language of higher dimensional class field theory over finite fields. Instead of the class group involving higher  $K$ -theory which was used in earlier work, see [KS] for example, we use a relative Chow group of zero cycles with modulus.

Let  $U$  be a smooth variety over a finite field  $k$ . The principle idea is to describe the abelian fundamental group  $\pi_1^{\text{ab}}(U)$  of  $U$  in terms of the Chow group  $C(X, D)$  with modulus  $D$ . For the definition of the latter we choose a compactification  $U \subset X$

with  $X$  normal and proper over  $k$  such that  $X \setminus U$  is the support of an effective Cartier divisor on  $X$  and  $D$  is an effective Cartier divisor with support  $|D|$  in  $X \setminus U$ . We define

$$C(X, D) = \text{Coker} \left( \bigoplus_{Z \subset U} k(Z)_D^\times \xrightarrow{\text{div}_Z} Z_0(U) \right),$$

where  $Z_0(U)$  is the group of zero-cycles on  $U$  and  $Z$  ranges over the integral closed curves on  $U$ . Here  $\text{div}_Z : k(Z)^\times \rightarrow Z_0(U)$  is the divisor map on the function field  $k(Z)$ . The group  $k(Z)_D^\times$  is the congruence subgroup of elements of  $k(Z)^\times$  which are congruent to 1 modulo the ideal  $I_D = \mathcal{O}_X(-D)$  at all infinite places of  $k(Z)$ .

More precisely let  $Z^N$  be the normalization of the closure of  $Z$  in  $X$  and let  $Z_\infty$  be the infinite places of  $k(Z)$ , i.e. the finite set of the closed points of  $Z^N$  which map to  $X \setminus U$ . We define

$$k(Z)_D^\times = \bigcap_{y \in Z_\infty} \text{Ker}(\mathcal{O}_{Z^N, y}^\times \rightarrow (\mathcal{O}_{Z^N, y}/I_D \mathcal{O}_{Z^N, y})^\times) \subset k(Z)^\times,$$

where  $\mathcal{O}_{Z^N, y}$  is the local ring of  $Z^N$  at  $y$ . Thus  $C(X, D)$  is an extension of the Chow group of zero-cycles of  $U$  which has been used repeatedly, see [ESV] [LW] [Ru]. It is also an extension of Suslin's singular homology  $H_0^{\text{sing}}(U, \mathbb{Z})$ , see [SV] and Remark 1.5 below. In case  $\dim(X) = 1$  it is the class group with modulus  $D$  used in classical class field theory.

We then introduce our class group of  $U$  as

$$C(U) := \varprojlim_D C(X, D),$$

where  $D$  runs through all effective Cartier divisors on  $X$  with  $|D| \subset X \setminus U$  and endow it with the inverse limit topology where  $C(X, D)$  is endowed with the discrete topology. We show that the topological group  $C(U)$  is independent of the compactification  $X$  of  $U$ , and construct a continuous map of topological groups, called the reciprocity map,

$$\rho_U : C(U) \rightarrow \pi_1^{\text{ab}}(U),$$

such that its composite with the natural map  $Z_0(U) \rightarrow C(U)$  is induced by the Frobenius maps  $\text{Frob}_x : \mathbb{Z} \rightarrow \pi_1^{\text{ab}}(U)$  for closed points  $x$  of  $U$ . The reciprocity map induces a continuous map

$$\rho_U^0 : C(U)^0 \rightarrow \pi_1^{\text{ab}}(U)^0.$$

Here  $\pi_1^{\text{ab}}(U)^0 = \text{Ker}(\pi_1^{\text{ab}}(U) \rightarrow \pi_1^{\text{ab}}(\text{Spec}(k)))$  and  $C(U)^0 = \text{Ker}(C(U) \xrightarrow{\deg} \mathbb{Z})$ , where  $\deg$  is induced by the degree map  $Z_0(U) \rightarrow \mathbb{Z}$ . Now our main result, see also Theorem 3.3, is the following.

**Theorem I** (Existence Theorem). *Assuming  $\text{ch}(k) \neq 2$ ,  $\rho_U^0$  is an isomorphism of topological groups.*

In case  $\dim(X) = 1$  the theorem is one of the main results in classical class field theory. In higher dimension a special case of Theorem I, describing only the tame quotient of  $\pi_1^{\text{ab}}(U)$ , is shown in [SS] (see also [Wi] and [KeSc]).

In [KS] an analog of Theorem I is shown with  $C(U)$  replaced by a different class group  $C^{KS}(U)$  explained below, which can be described in terms of higher local fields associated to chains of subvarieties on a compactification  $X$  of  $U$ . Recall  $C(U)$  is defined only in terms of points and curves on  $U$ . There is a factorization of the reciprocity map

$$C(U) \rightarrow C^{KS}(U) \rightarrow \pi_1^{\text{ab}}(U)$$

and the main result of [KS] over a finite field, see also [Ra, Thm. 6.2], is a direct consequence of our Theorem I.

Using ramification theory in local class field theory, Theorem I implies.

**Corollary II.** *Assume  $\text{ch}(k) \neq 2$ . For an effective divisor  $D$  on  $X$  with  $|D| \subset X \setminus U$ ,  $\rho_U$  induces an isomorphism of finite groups*

$$C(X, D)^0 \xrightarrow{\sim} \pi_1^{\text{ab}}(X, D)^0,$$

where  $\pi_1^{\text{ab}}(X, D)$  is the quotient of  $\pi_1^{\text{ab}}(X)$  which classifies abelian étale coverings of  $U$  with ramification over  $X \setminus U$  bounded by the divisor  $D$ .

The finiteness of  $C(X, D)^0$  is equivalent to the rank one case of Deligne's finiteness theorem (see [EK, Thm. 8.1]). Our arguments yield an alternative proof of this finiteness result.

**Ramification theory.** The Pontryagin dual  $\text{fil}_D H^1(U)$  of  $\pi_1^{\text{ab}}(X, D)$  is the group of continuous characters  $\chi : \pi_1^{\text{ab}}(U) \rightarrow \mathbb{Q}/\mathbb{Z}$  such that for any integral curve  $Z \subset U$ , its restriction  $\chi|_Z : \pi_1^{\text{ab}}(Z) \rightarrow \mathbb{Q}/\mathbb{Z}$  satisfies the following equality of Cartier divisors on  $Z^N$ :

$$\sum_{y \in Z_\infty} \text{ar}_y(\chi|_Z)[y] \leq \psi_Z^* D,$$

where  $\text{ar}_y(\chi|_Z) \in \mathbb{Z}_{\geq 0}$  is the Artin conductor of  $\chi|_Z$  at  $y \in Z_\infty$  and  $\psi_Z^* D$  is the pullback of  $D$  by the natural map  $\psi_Z : Z^N \rightarrow X$  (see Definition 2.8).

Our proof of Theorem I depends in an essential way on ramification theory due to Kato [Ka1] and its variant by Matsuda [Ma]. Let  $K_\lambda$  be the henselization of  $K = k(U)$  at the generic point  $\lambda$  of an irreducible component  $C_\lambda$  of  $X \setminus U$  and let  $H^1(K_\lambda)$  be the group of continuous characters  $G_{K_\lambda} \rightarrow \mathbb{Q}/\mathbb{Z}$ , where  $G_{K_\lambda}$  is the absolute Galois group of  $K_\lambda$ . They introduced a ramification filtration  $\text{fil}_m H^1(K_\lambda)$  ( $m \in \mathbb{Z}_{\geq 0}$ ) on  $H^1(K_\lambda)$  which generalizes the ramification filtration for local fields with perfect residue fields (see [Se1]), and defined a natural injective map

$$(0.1) \quad \text{rar}_{K_\lambda} : \text{fil}_m H^1(K_\lambda) / \text{fil}_{m-1} H^1(K_\lambda) \hookrightarrow \Omega_X^1(mC_\lambda) \otimes_{\mathcal{O}_X} k(C_\lambda) \quad (m > 1)$$

which we call refined Artin conductor (indeed what Kato originally defined is refined Swan conductor and we use a variant for Artin conductor introduced by Matsuda), where  $k(C_\lambda)$  is the function field of  $C_\lambda$ . In case  $C = X \setminus U$  is a simple normal crossing divisor on smooth  $X$ , results from the ramification theory imply

$$\text{fil}_D H^1(U) = \text{Ker} \left( H^1(U) \rightarrow \bigoplus_{\lambda \in I} H^1(K_\lambda) / \text{fil}_{m_\lambda} H^1(K_\lambda) \right).$$

Here  $H^1(U)$  denotes the group of continuous characters  $\chi : \pi_1^{\text{ab}}(U) \rightarrow \mathbb{Q}/\mathbb{Z}$ ,  $I$  is the set of the generic points  $\lambda$  of  $C$  and  $m_\lambda$  is the multiplicity of  $C_\lambda$  in  $D$ .

Now the basic strategy of the proof of Theorem I is as follows (see §3 for the details). By an argument due to Wiesend we are allowed to replace  $X$  by an alteration  $f : X' \rightarrow X$  and  $U$  by a smooth open  $U' \subset f^{-1}(U)$ . Then a Lefschetz theorem for  $\pi_1^{\text{ab}}(X, D)$  (cf. [KeS]) reduces the proof to the case where  $X$  is a smooth projective surface and  $C = X \setminus U$  is a simple normal crossing divisor. The proof then proceeds by induction on the multiplicity of  $D$  reducing to the tame case  $D = C$ . A key point is the construction of a natural map, which we call the *cycle conductor*, defined for Cartier divisors  $D$  such that  $D \geq 2C$ :

$$\text{cc}_{X,D} : C(X, D)^\vee := \text{Hom}(C(X, D), \mathbb{Q}/\mathbb{Z}) \rightarrow H^0(C, \Omega_X^1(D + \Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C),$$

where  $\Xi \subset X$  is an auxiliary effective Cartier divisor independent of  $D$ . It satisfies

$$\text{Ker}(\text{cc}_{X,D}) = C(X, D - C)^\vee \subset C(X, D)^\vee,$$

and the following diagram

$$\begin{array}{ccccc}
\mathrm{fil}_D H^1(U) & \longrightarrow & \mathrm{fil}_{m_\lambda} H^1(K_\lambda) & \xrightarrow{\mathrm{rar}_{K_\lambda}} & \Omega_X^1(D) \otimes_{\mathcal{O}_X} k(C_\lambda) \\
\downarrow \Psi_{X,D} & & & & \uparrow \\
C(X, D)^\vee & \xrightarrow{\mathrm{cc}_{X,D}} & H^0(C, \Omega_X^1(D + \Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C) & & 
\end{array}$$

commutes. Here  $\Psi_{X,D}$  is the dual of the reciprocity map  $C(X, D) \rightarrow \pi_1^{\mathrm{ab}}(X, D)$  induced by  $\rho_U$ . Therefore we consider the cycle conductor  $\mathrm{cc}_{X,D}$  as a cycle theoretic avatar of the refined Artin conductor of Kato and Matsuda.

By duality, the definition of cycle conductors is reduced to the construction of a natural map

$$(0.2) \quad \phi_{X,D} : H^1(C, \Omega_X^1(-D + C - \Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C) \rightarrow C(X, D)$$

such that the following sequence is exact

$$H^1(C, \Omega_X^1(-D + C - \Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C) \xrightarrow{\phi_{X,D}} C(X, D) \rightarrow C(X, D - C) \rightarrow 0.$$

In fact it turns out that such a map exists even over a general perfect field  $k$  at least after replacing  $C(X, D)$  by a partial  $p$ -adic completion (see Theorem 4.5 and Remark 4.6 for details).

**General base fields.** Let now  $k$  be an arbitrary perfect field of characteristic  $p > 0$  and let  $U$  be a smooth variety of dimension  $d$  over  $k$  as above. The previous result suggests the following question. Note that it has a positive answer for  $k$  finite by Theorem I even without  $p$ -adic completion.

**Question III.** *Does the natural map*

$$C(U) \rightarrow C^{KS}(U)$$

*to Kato–Saito class group over a general perfect field  $k$  become an isomorphism after  $p$ -adic completion?*

Recall that the Kato–Saito class group is defined in terms of Nisnevich cohomology groups

$$C^{KS}(U) = \varprojlim_D H^d(X_{\mathrm{Nis}}, \mathcal{K}_d^M(X, D)).$$

Here  $\mathcal{K}_d^M(X, D)$  is the relative Milnor  $K$ -sheaf of [KS] and  $D$  runs through all effective Cartier divisors on  $X$  with  $|D| \subset X \setminus U$ .

**Skeleton sheaves.** Let again  $X$  be a normal variety over the finite field  $k$  and let  $C$  be as above. Consider a family  $(V_Z)_Z$ , where  $Z$  runs through all closed integral curves on  $U$  and where  $V_Z$  is a semi-simple lisse  $\overline{\mathbb{Q}}_\ell$ -sheaf on the normalization  $\tilde{Z}$  of  $Z$ . We say that the family  $(V_Z)_Z$  is compatible if for two different curves  $Z_1, Z_2$  the sheaves  $V_{Z_1}$  and  $V_{Z_2}$  become isomorphic up to semi-simplification after pullback to the finite scheme  $(\tilde{Z}_1 \times_X \tilde{Z}_2)_{\mathrm{red}}$ . Such compatible families are also called *skeleton sheaves* and have been studied by Deligne and Drinfeld, see [EK].

We say that a skeleton sheaf  $(V_Z)_Z$  has bounded ramification if there exists an effective Cartier divisor  $D$  with  $|D| \subset C$  and such that

$$\sum_{y \in Z_\infty} \mathrm{ar}_y(V_Z)[y] \leq \psi_Z^* D$$

for all integral curves  $Z$  on  $U$ .

**Question IV** (Deligne). *Does any skeleton sheaf  $(V_Z)_Z$  with bounded ramification come from a lisse  $\overline{\mathbb{Q}}_\ell$ -sheaf  $V$  on  $U$ , i.e. is the semi-simplification of  $V|_Z$  isomorphic to  $V_Z$  for all curves  $Z$ ?*

Combining Theorem I with classical global class field theory we obtain:

**Corollary V.** *Question IV has a positive answer in rank one.*

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We give an overview of the content of the paper.

In §1 we introduce a class group  $W(U)$  studied by Wiesend [Wi]. We define some filtrations on  $W(U)$  and explain their basic properties. Our class group  $C(X, D)$  can be defined as a quotient of  $W(U)$  by a certain filtration. We also introduce a tool to produce relations in  $W(U)$ .

In §2 we review some results on ramification theory. The first subsection treats local ramification theory for henselian discrete valuation fields whose residue fields are not necessarily perfect, originally due to Kato [Ka1], [Ka2] and [Ka3]. The refined Artin conductor, see (0.1), is introduced, which plays a key role in this paper. In the second subsection, some implications on global ramification theory are given.

In §3 the reciprocity map  $\rho_U$  is defined and the Existence Theorem is stated. The basic strategy of the proof of the Existence Theorem is explained. We explain an argument due to Wiesend, which allows us to replace  $X$  by an alteration. We reduce the proof to the case  $\dim(X) = 2$  by using the Lefschetz theorem for abelian fundamental groups allowing ramification along some divisor, which is proved in [KeS].

In §4 we start the construction of the map  $\phi_{X,D}$ , see (0.2). It is the dual of the cycle conductor which is a key ingredient of the proof of the Existence Theorem. The description of local components of  $\phi_{X,D}$  is given, depending a priori on a choice of local parameters. The independence of the choice is deduced from a technical key lemma from §6. Two key theorems are stated. The first theorem asserts that the collection of the local components induces the desired map  $\phi_{X,D}$ . The second theorem states a compatibility of  $\phi_{X,D}$  with the refined Artin conductor. The proof of the Existence Theorem is completed in §5 using these key theorems.

The key theorems are proved in §7 and §8 assuming the technical key lemmas stated in §6. The proof of these lemma occupies the later sections §9 through §13. The tool to produce relations in  $W(U)$  introduced in §1 will play a basic role in the proof.

There is work related to our main results by H. Russell involving a geometric method based on his joint work with K. Kato on Albanese varieties with modulus.

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## 1. WIESEND CLASS GROUP AND FILTRATIONS

In the whole paper we fix a perfect field  $k$  with  $\text{ch}(k) = p > 0$ . At many places we have to assume  $p \neq 2$ . Let  $X$  be a proper normal scheme over  $k$  and  $C$  be the support of an effective Cartier divisor on  $X$  and put  $U = X \setminus C$ . Note that  $C$  is a reduced closed subscheme of pure codimension one,

**Definition 1.1.** Let  $Z_1(X)^+$  be the monoid of effective 1-cycles on  $X$  and

$$Z_1(X, C)^+ \subset Z_1(X)^+$$

be the submonoid of the cycles  $Z$  such that none of the prime components of  $Z$  is contained in  $C$ . Take

$$Z = \sum_{1 \leq i \leq r} n_i Z_i \in Z_1(X)^+,$$

where  $Z_1, \dots, Z_r$  are the prime components of  $Z$  and  $n_i \in \mathbb{Z}_{\geq 0}$ . We write

$$k(Z)^\times = k(Z_1)^\times \oplus \dots \oplus k(Z_r)^\times,$$

$$|Z| = \bigcup_{1 \leq i \leq r} Z_i \subset X, \quad I_Z = \prod_{1 \leq i \leq r} (I_{Z_i})^{n_i} \subset \mathcal{O}_X,$$

where  $I_{Z_i} \subset \mathcal{O}_X$  is the ideal of  $Z_i$ . We say  $Z$  is reduced if  $n_i = 1$  for all  $1 \leq i \leq r$  and integral if it is reduced and  $r = 1$ . We say  $Z$  intersects  $C$  transversally at  $x \in X$  (denoted by  $Z \pitchfork C$  at  $x$ ) if  $|Z|$  and  $C$  are regular at  $x$  and the intersection multiplicity

$$(Z, C)_x := \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/I_Z + I_C) = 1.$$

We say  $Z$  intersects  $C$  transversally (denoted by  $Z \pitchfork C$ ) if  $Z \pitchfork C$  at all  $x \in Z \cap C$ .

**Definition 1.2.** For  $Z \in Z_1(X, C)^+$ , let  $\psi_Z : Z^N \rightarrow |Z|$  be the normalization and put

$$Z_\infty = \{y \in Z^N \mid \psi_Z(x) \in |Z| \cap C\}.$$

For  $y \in Z_\infty$ , let  $k(Z)_y$  be the henselization of  $k(Z)$  at  $y$  and put

$$k(Z)_\infty = \prod_{y \in Z_\infty} k(Z)_y \quad \text{and} \quad k(Z)_\infty^\times = \prod_{y \in Z_\infty} k(Z)_y^\times.$$

The Wiesend class group of  $U$  is defined as

$$(1.1) \quad W(U) = \text{Coker} \left( \bigoplus_{Z \subset X} k(Z)^\times \xrightarrow{\delta} \bigoplus_{Z \subset X} k(Z)_\infty^\times \oplus Z_0(U) \right),$$

where  $Z$  ranges over the integral elements of  $Z_1(X, C)^+$  [Wi], [KeSc]. Here we map  $k(Z)^\times$  diagonally in  $k(Z)_\infty^\times$  and  $k(Z)^\times \rightarrow Z_0(U)$  is the composite map

$$\kappa(Z)^\times \xrightarrow{\text{div}_{Z \cap U}} Z_0(Z \cap U) \hookrightarrow Z_0(U).$$

Obviously  $W(U)$  depends only on  $U$ , i.e. is independent of  $(X, C)$  such that  $U = X \setminus C$ .

For a morphism  $f : U' \rightarrow U$  of smooth varieties there is an canonical induced morphism

$$(1.2) \quad f_* : W(U') \rightarrow W(U),$$

see [Wi] and [KeSc, Sec. 7]. If  $f$  is finite we also speak of the norm map and write  $N_f$  for  $f_*$ .

**Definition 1.3.** Let  $Z \in Z_1(X, C)^+$ .

- (1) Assume  $Z$  integral. For  $x \in Z \cap C$ , we have the natural map

$$\{ \}_{Z,x} : \bigoplus_{y \in \psi_Z^{-1}(x)} k(Z)_y^\times \rightarrow W(U)$$

where  $\psi_Z : Z^N \rightarrow Z$  is the normalization. Taking the sum of these maps for  $x \in Z \cap C$ , we get

$$\{ \}_Z : k(Z)_\infty^\times \rightarrow W(U).$$

- (2) In general we write  $Z = \sum_{i \in I} e_i Z_i$  where  $\{Z_i\}_{i \in I}$  are the prime components of  $Z$  and  $e_i \in \mathbb{Z}_{\geq 0}$ , and define

$$\{ \}_{Z,x} = \sum_{i \in I} e_i \{ \}_{Z_i,x} ; \quad \bigoplus_{y \in \psi_Z^{-1}(x)} k(Z)_y^\times \rightarrow W(U),$$

$$\{ \}_Z = \sum_{i \in I} e_i \{ \}_{Z_i} ; \quad k(Z)_\infty^\times \rightarrow W(U).$$

where  $\psi_Z : Z^N \rightarrow |Z|$  is the normalization.

Let the notation be as in Definition 1.3 and  $Z \in Z_1(X, C)^+$ . Write  $\mathcal{O}_Z = \mathcal{O}_X/I_Z$  and  $\mathcal{O}_{Z,x}^h$  for the henselization of  $\mathcal{O}_{Z,x}$  for  $x \in |Z|$ . We also write

$$\mathcal{O}_{Z, C \cap Z}^h = \prod_{x \in Z \cap C} \mathcal{O}_{Z,x}^h, \quad \mathcal{O}_{Z^N, C \cap Z}^h = \mathcal{O}_{Z, C \cap Z}^h \otimes_{\mathcal{O}_Z} \mathcal{O}_{Z^N} = \prod_{y \in \psi_Z^{-1}(Z \cap C)} \mathcal{O}_{Z^N, y}^h.$$

We have the natural maps

$$\mathcal{O}_{Z, C \cap Z}^h \rightarrow \mathcal{O}_{Z^N, C \cap Z}^h \hookrightarrow k(Z)_\infty$$

**Definition 1.4.** Let  $D$  be an effective Cartier divisor such that  $|D| \subset C$  and let  $I_D = \mathcal{O}_X(-D)$  be the ideal sheaf of  $D$ .

- (1) We define  $F^{(D)}W(X, C) \subset W(U)$  as the subgroup generated by

$$\{1 + I_D \mathcal{O}_{Z, C \cap Z}^h\}_Z$$

for all  $Z \in Z_1(X, C)^+$ .

- (2) We define  $\widehat{F}^{(D)}W(X, C) \subset W(U)$  as the subgroup generated by

$$\{1 + I_D \mathcal{O}_{Z^N, C \cap Z}^h\}_Z$$

for all  $Z \in Z_1(X, C)^+$ . Note  $F^{(D)}W(X, C) \subset \widehat{F}^{(D)}W(X, C)$ .

- (3) We define  $\widehat{F}^{(1)}W(U) \subset W(U)$  as the subgroup generated by

$$\{1 + \mathfrak{m} \mathcal{O}_{Z^N, C \cap Z}^h\}_Z$$

for all  $Z \in Z_1(X, C)^+$ , where  $\mathfrak{m}$  is the Jacobson radical of  $\mathcal{O}_{Z^N, C \cap Z}^h$ . Note that  $\widehat{F}^{(1)}W(U)$  depends only on  $U$  and  $\widehat{F}^{(D)}W(X, C) \subset \widehat{F}^{(1)}W(U)$  if  $|D| = C$  and  $\widehat{F}^{(1)}W(U)/F^{(D)}W(X, C)$  is  $p$ -primary torsion.

- (4) For a dense open subset  $V \subset X$  containing the generic points of  $C$ , we define

$$F_{\mathfrak{m}V}^{(D)}W(X, C) = \sum_{G, x} \{1 + \mathcal{O}_{G, G \cap C}(-D)\}_{G, x} \subset F^{(D)}W(X, C),$$

where  $x$  ranges over the regular closed points of  $C \cap V$  and  $G$  ranges over  $Z_1(X, C)^+$  such that  $G \mathfrak{m} C$  at  $x$ . In case  $V = X$  we simply denote  $F_{\mathfrak{m}V}^{(D)}W(X, C) = F_{\mathfrak{m}}^{(D)}W(X, C)$ .

*Remark 1.5.* It is shown in [Sc, Thm 3.1] that there is a natural isomorphism

$$W(U)/\hat{F}^{(1)}W(U) \simeq H_0^{\text{sing}}(U, \mathbb{Z}),$$

where the right hand side is Suslin's singular homology.

**Definition 1.6.** Under the notation of Definition 1.4, we put

$$C(X, D) = W(U)/\hat{F}^{(D)}W(X, C).$$

By the weak approximation theorem, we have an isomorphism

$$C(X, D) \simeq \text{Coker} \left( \bigoplus_{Z \subset X} k(Z)_D^\times \rightarrow Z_0(U) \right),$$

where  $Z$  ranges over the integral elements of  $Z_1(X, C)^+$ , and

$$\begin{aligned} k(Z)^\times \supset k(Z)_D^\times &= \text{Ker}(k(Z)^\times \rightarrow \prod_{y \in Z_\infty} k(Z)_y^\times / 1 + I_D \mathcal{O}_{Z^N, y}) \\ &= \bigcap_{y \in Z_\infty} \text{Ker}(\mathcal{O}_{Z^N, y}^\times \rightarrow (\mathcal{O}_{Z^N, y} / I_D \mathcal{O}_{Z^N, y})^\times). \end{aligned}$$

Thus  $C(X, D)$  is an extension of the Chow group of zero-cycles of  $U$ .

**Lemma 1.7.** *Let  $f : X' \rightarrow X$  be a morphism with  $f(X') \cap U \neq \emptyset$  and let  $D$  be an effective Cartier divisor on  $X$  with  $|D| \subset C$ . Set  $U' = f^{-1}(U)$  and  $D' = f^*D$ . Then the pushforward (1.2) satisfies  $f_*(\hat{F}^{(D')}W(X', C')) \subset \hat{F}^{(D)}W(X, C)$ .*

*Proof.* This is a direct consequence of the definition and standard properties of the norm map for local fields.  $\square$

In what follows we assume  $\dim(X) = 2$ .

**Definition 1.8.** Let  $X$  be a projective smooth surface over  $k$ . Let  $\text{Div}(X)^+$  be the monoid of effective Cartier divisors on  $X$  and  $\text{Div}(X, C)^+ \subset \text{Div}(X)^+$  be the submonoid of such Cartier divisors  $D$  that none of the prime components of  $D$  is contained in  $C$ . Then  $Z_1(X)^+$  coincides with  $\text{Div}(X)^+$  and  $Z_1(X, C)^+$  coincides with  $\text{Div}(X, C)^+$ .

**Definition 1.9.** Let  $\mathcal{C}$  be the category of triples  $(X, C)$ , where

- $X$  is a projective smooth surface over  $k$ ,
- $C$  is a reduced Cartier divisor on  $X$ .

A morphism  $f : (X', C') \rightarrow (X, C)$  in  $\mathcal{C}$  is a surjective map  $f : X' \rightarrow X$  of schemes such that  $C' = f^{-1}(C)_{\text{red}}$ . For  $f$  as above and for  $D \in \text{Div}(X)^+$ , we let  $f^*D \in \text{Div}(X)^+$  be the pullback of  $D$  as a Cartier divisor. Let  $\mathcal{C}_X \subset \mathcal{C}$  be the subcategory of the objects over  $X$ .

**Definition 1.10.** Let  $X = (X, C)$  be in  $\mathcal{C}$ .

- (1) Let  $\hat{\mathcal{B}}_X \subset \mathcal{C}_X$  be the subcategory of the object  $(\tilde{X}, \tilde{C})$ , where  $g : \tilde{X} \rightarrow X$  is the composite of successive blowups at closed points.
- (2) Let  $\mathcal{B}_X \subset \hat{\mathcal{B}}_X$  be the subcategory of the object  $(\tilde{X}, \tilde{C})$ , where  $g : \tilde{X} \rightarrow X$  is the composite of successive blowups at closed points of regular loci of preimages of  $C$ .

**Lemma 1.11.** *Let  $X = (X, C)$  be in  $\mathcal{C}$  and  $D \in \text{Div}(X)^+$  such that  $|D| = C$ . For  $g : (\tilde{X}, \tilde{C}) \rightarrow (X, C)$  in  $\hat{\mathcal{B}}_X$ , we have*

$$F^{(D)}W(X, C) \subset F^{(g^*D)}W(\tilde{X}, \tilde{C}) \subset \hat{F}^{(D)}W(X, C).$$

We have

$$\widehat{F}^{(D)}W(X, C) = \lim_{g: \tilde{X} \rightarrow X} F^{(g^*D)}W(\tilde{X}, \tilde{C}),$$

where  $g: (\tilde{X}, \tilde{C}) \rightarrow (X, C)$  ranges over  $\widehat{\mathcal{B}}_X$ .

*Proof.* Take integral  $Z \in \text{Div}(X, C)^+$  and let  $Z' \in \text{Div}(\tilde{X}, \tilde{C})^+$  be its proper transform. Then  $Z'$  is finite over  $Z$  and we have

$$\mathcal{O}_{Z, C \cap Z}^h \subset \mathcal{O}_{Z', \tilde{C} \cap Z'}^h \subset \mathcal{O}_{Z^N, C \cap Z}^h.$$

The first assertion follows from these facts. The second assertion follows from the fact that for any integral  $Z \in \text{Div}(X, C)^+$ , there is  $g: \tilde{X} \rightarrow X$  in  $\widehat{\mathcal{B}}_X$  such that the proper transform of  $Z$  in  $\tilde{X}$  is regular.  $\square$

**Definition 1.12.** For  $X = (X, C)$  and  $D$  as in Lemma 1.11, we put

$$F_{\mathcal{B}}^{(D)}W(X, C) = \lim_{g: \tilde{X} \rightarrow X} F^{(g^*D)}W(\tilde{X}, \tilde{C}),$$

where  $g: (\tilde{X}, \tilde{C}) \rightarrow (X, C)$  ranges over  $\mathcal{B}_X$ . Lemma 1.11 implies

$$F^{(D)}W(X, C) \subset F_{\mathcal{B}}^{(D)}W(X, C) \subset \widehat{F}^{(D)}W(X, C).$$

For  $g: (\tilde{X}, \tilde{C}) \rightarrow (X, C)$  in  $\mathcal{B}_X$ , we have

$$(1.3) \quad F_{\mathcal{B}}^{(g^*D)}W(\tilde{X}, \tilde{C}) = F_{\mathcal{B}}^{(D)}W(X, C).$$

*Remark 1.13.* For  $Z \in \text{Div}(X, C)^+$ , we have an isomorphism

$$\mathcal{O}_{Z, C \cap Z} \otimes_{\mathcal{O}_X} \mathcal{O}_C \simeq \prod_{x \in Z \cap C} \mathcal{O}_{Z, x}^h \otimes_{\mathcal{O}_X} \mathcal{O}_C,$$

Hence we have an isomorphism

$$\frac{1 + I_D \mathcal{O}_{Z, C \cap Z}}{1 + I_{D+C} \mathcal{O}_{Z, C \cap Z}} \simeq \bigoplus_{x \in Z \cap C} \frac{1 + I_D \mathcal{O}_{Z, x}^h}{1 + I_{D+C} \mathcal{O}_{Z, x}^h}$$

and we have

$$\sum_{x \in Z \cap C} \{1 + I_D \mathcal{O}_{Z, x}^h\}_{Z, x} = \{1 + I_D \mathcal{O}_{Z, C \cap Z}\}_Z + \{1 + I_{D+C} \mathcal{O}_{Z, C \cap Z}^h\}_Z.$$

Let  $X = (X, C)$  be in  $\mathcal{C}$ . We introduce a tool to produce relations in  $W(U)$  by using symbols in the Milnor  $K$ -group  $K_2^M(k(X))$  of the function field  $k(X)$  of  $X$ .

**Lemma 1.14.** Let  $a, b \in k(X)^\times$  and write as divisors

$$\text{div}_X(a) = Z_a^+ - Z_a^- + W_a, \quad \text{div}_X(b) = Z_b^+ - Z_b^- + W_b,$$

where  $Z_a^+, Z_a^-, Z_b^+, Z_b^-$  are in  $\text{Div}(X, C)^+$  and  $W_a, W_b$  have support in  $C$ . Assume that  $Z_a^+, Z_a^-, Z_b^+, Z_b^-$  have no common component.

Then

$$\partial\{a, b\} := \{a|_{Z_b^+}\}_{Z_b^+} - \{a|_{Z_b^-}\}_{Z_b^-} - \{b|_{Z_a^+}\}_{Z_a^+} + \{b|_{Z_a^-}\}_{Z_a^-},$$

vanishes in  $W(U)$ . Here  $a|_{Z_b^\pm} \in k(Z_b^\pm)_\infty^\times$  (resp.  $b|_{Z_a^\pm} \in k(Z_a^\pm)_\infty^\times$ ) is the image of  $a$  (resp.  $b$ ).

*Proof.* Let  $\alpha^+ \in k(Z_b^+)^\times$ ,  $\alpha^- \in k(Z_b^-)^\times$  be the restrictions of  $a$  and  $\beta^+ \in k(Z_a^+)^\times$ ,  $\beta^- \in k(Z_a^-)^\times$  be the restrictions of  $b$  (see Definition 1.1). Obviously, the elements

$$\delta(\alpha^+), \delta(\alpha^-), \delta(\beta^+), \delta(\beta^-)$$

map to zero in  $W(U)$ , with  $\delta$  as in (1.1). In order to finish the proof of the lemma it suffices to show

$$\partial\{a, b\} = \delta(\alpha^+) - \delta(\alpha^-) - \delta(\beta^+) + \delta(\beta^-) \in \bigoplus_{Z \subset X} k(Z)_\infty^\times \oplus Z_0(U),$$

i.e. that the contributions of the right hand side at any closed point  $x \in U$  cancel out. This is a consequence of the Gersten complex for  $K$ -theory

$$K_2(k(X)) \rightarrow \bigoplus_{y \in \text{Spec}(\mathcal{O}_{U,x})^{(1)}} K_1(y) \rightarrow K_0(x) = \mathbb{Z}.$$

□

## 2. REVIEW OF RAMIFICATION THEORY

**2.1. Local ramification theory.** In this subsection  $K$  denotes a henselian discrete valuation field of  $\text{ch}(K) = p > 0$  with ring of integers  $\mathcal{O}_K$  and residue field  $E$ . Let  $\pi$  be a prime element of  $\mathcal{O}_K$  and  $\mathfrak{m}_K = (\pi) \subset \mathcal{O}_K$  be the maximal ideal. By the Artin–Schreier–Witt theory, we have a natural isomorphism for  $s \in \mathbb{Z}_{\geq 1}$ ,

$$(2.1) \quad \delta_s : W_s(K)/(1-F)W_s(K) \xrightarrow{\cong} H^1(K, \mathbb{Z}/p^s\mathbb{Z}),$$

where  $W_s(K)$  is the ring of Witt vectors of length  $s$  and  $F$  is the Frobenius. We have the Brylinski–Kato filtration

$$\text{fil}_m^{\log} W_s(K) = \{(a_{s-1}, \dots, a_1, a_0) \in W_s(K) \mid p^i v_K(a_i) \geq -m\},$$

where  $v_K$  is the normalized valuation of  $K$ . In this paper we use its non-log version introduced by Matsuda [Ma]:

$$\text{fil}_m W_s(K) = \text{fil}_{m-1}^{\log} W_s(K) + V^{s-s'} \text{fil}_m^{\log} W_{s'}(K),$$

where  $s' = \min\{s, \text{ord}_p(m)\}$  and  $V : W_{s-1}(K) \rightarrow W_s(K)$  is the Verschiebung. We define ramification filtrations on  $H^1(K) := H^1(K, \mathbb{Q}/\mathbb{Z})$  as

$$\text{fil}_m^{\log} H^1(K) = H^1(K)\{p'\} \oplus \bigcup_{s \geq 1} \delta_s(\text{fil}_m^{\log} W_s(K)) \quad (m \geq 0),$$

$$\text{fil}_m H^1(K) = H^1(K)\{p'\} \oplus \bigcup_{s \geq 1} \delta_s(\text{fil}_m W_s(K)) \quad (m \geq 1),$$

where  $H^1(K)\{p'\}$  is the prime-to- $p$  part of  $H^1(K)$ . We note that  $\text{fil}_m H^1(K)$  is shifted by one from Matsuda’s filtration [Ma, Def.3.1.1]. We also let  $\text{fil}_0 H^1(K)$  be the subgroup of all unramified Galois characters.

**Definition 2.1.** For  $\chi \in H^1(K)$  we denote the minimal  $m$  with  $\chi \in \text{fil}_m H^1(K)$  by  $\text{ar}_K(\chi)$  and call it the Artin conductor of  $\chi$ .

In case the field  $E$  is perfect this definition coincides with the classical definition, see [Ka2, Prop. 6.8].

We have the following fact (cf. [Ka2] and [Ma]).

**Lemma 2.2.** (1)  $\text{fil}_1 H^1(K)$  is the subgroup of tamely ramified characters.

(2)  $\text{fil}_m H^1(K_\lambda) \subset \text{fil}_m^{\log} H^1(K) \subset \text{fil}_{m+1} H^1(K)$ .

(3)  $\text{fil}_m H^1(K) = \text{fil}_{m-1}^{\log} H^1(K)$  if  $(m, p) = 1$ .

The structure of graded quotients:

$$\mathrm{gr}_m H^1(K) = \mathrm{fil}_m H^1(K) / \mathrm{fil}_{m-1} H^1(K) \quad (m > 1)$$

are described as follows. Let  $\Omega_K^1$  be the absolute Kähler differential module and put

$$\mathrm{fil}_m \Omega_K^1 = \mathfrak{m}_K^{-m} \otimes_{\mathcal{O}_K} \Omega_K^1.$$

We have an isomorphism

$$(2.2) \quad \mathrm{gr}_m \Omega_K^1 = \mathrm{fil}_m \Omega_K^1 / \mathrm{fil}_{m-1} \Omega_K^1 \simeq \mathfrak{m}_K^{-m} \Omega_K^1 \otimes_{\mathcal{O}_K} E.$$

We have the maps

$$F^s d : W_s(K) \rightarrow \Omega_K^1 ; (a_{s-1}, \dots, a_1, a_0) \rightarrow \sum_{i=0}^{s-1} a_i^{p^i-1} da_i.$$

and one can check  $F^s d(\mathrm{fil}_m W_s(K)) \subset \mathrm{fil}_m \Omega_K^1$ .

**Theorem 2.3.** ([Ma]) Assume  $m > 1$ .

- (1) The maps  $F^s d$  factor through  $\delta_s$  and induce a natural map

$$\mathrm{fil}_m H^1(K) \rightarrow \mathrm{fil}_m \Omega_K^1$$

which induces an injective map (called the refined Artin conductor for  $K$ )

$$(2.3) \quad \mathrm{rar}_K : \mathrm{gr}_m H^1(K) \hookrightarrow \mathrm{gr}_m \Omega_K^1.$$

- (2) If the residue field of  $K$  is perfect the map (2.3) is surjective.

**Definition 2.4.** Let  $K$  be as before and  $K_N^M(K)$  be the  $N$ -th Milnor  $K$ -group of  $K$ . For an integer  $m \geq 1$ , we define  $V^m K_N^M(K) \subset K_N^M(K)$  as a subgroup generated by the elements of the form

$$\{1 + a, b_1, \dots, b_{N-1}\} \quad \text{and} \quad \{1 + a\pi, b_1, \dots, b_{N-2}, \pi\},$$

where  $a \in \mathfrak{m}_K^m$  and  $b_1, \dots, b_N \in \mathcal{O}_K^\times$ .

The following lemma is proved by a similar argument as the proof of [BK, Lem.(4.2)].

**Lemma 2.5.** There is a canonical surjective map

$$\rho_K^m : \mathfrak{m}_K^{m-1} \Omega_K^{N-1} \otimes_{\mathcal{O}_K} E \rightarrow V^{m-1} K_N^M(K) / V^m K_N^M(K),$$

such that

$$\rho_K^m(\mathrm{adb}_1 \wedge \dots \wedge \mathrm{db}_{N-1}) = \{1 + ab_1 \dots b_{N-1}, b_1, \dots, b_{N-1}\}.$$

where  $a \in \mathfrak{m}_K^{m-1}$  and  $b_1, \dots, b_N \in \mathcal{O}_K$ .

Now we assume  $K$  is an  $N$ -dimensional local field in the sense of [Ka1], namely there is a sequence of fields  $k_0, \dots, k_N$  such that  $k_0$  is finite,  $k_N = K$ , and for  $1 \leq i \leq N$ ,  $k_i$  is a henselian discrete valuation field with residue field  $k_{i-1}$ . Then Kato [Ka2] defined a canonical map

$$(2.4) \quad \Psi_K : H^1(K) \rightarrow \mathrm{Hom}(K_N^M(K), \mathbb{Q}/\mathbb{Z}),$$

which satisfies the following conditions (cf. [Ka3, §3.5]):

- (i) For  $m \in \mathbb{Z}_{\geq 1}$  and  $\chi \in H^1(K)$ , we have an equivalence of conditions:

$$\chi \in \mathrm{fil}_m H^1(K) \iff \Psi_K(\chi)(V^m K_N^M(K)) = 0.$$

(ii) The following diagram is commutative

$$\begin{array}{ccc} \mathrm{fil}_m H^1(K) & \xrightarrow{\Psi_k} & \mathrm{Hom}(K_N^M(K)/V^m K_N^M(K), \mathbb{Q}/\mathbb{Z}) \\ \mathrm{rar}_K \downarrow & & \downarrow (\rho_K^m)^\vee \\ \mathfrak{m}_K^{-m} \Omega_{\mathcal{O}_K}^1 \otimes_{\mathcal{O}_K} E & \xrightarrow{\tau} & \mathrm{Hom}(\mathfrak{m}_K^{m-1} \Omega_{\mathcal{O}_K}^{N-1} \otimes_{\mathcal{O}_K} E, \mathbb{Q}/\mathbb{Z}) \end{array}$$

where the right vertical map is induced by  $\rho_K^m$  and  $\tau$  is induced by the pairing

$$\begin{aligned} \mathfrak{m}_K^{-m} \Omega_{\mathcal{O}_K}^1 \otimes_{\mathcal{O}_K} E \times \mathfrak{m}_K^{m-1} \Omega_{\mathcal{O}_K}^{N-1} \otimes_{\mathcal{O}_K} E &\rightarrow \mathfrak{m}_K^{-1} \Omega_{\mathcal{O}_K}^N \otimes_{\mathcal{O}_K} E \xrightarrow{\mathrm{Res}} \Omega_E^{N-1} \\ &\xrightarrow{\mathrm{Res}_{E/\mathbb{F}_p}} \mathbb{F}_p \simeq \mathbb{Z}/p\mathbb{Z}, \end{aligned}$$

where  $\mathrm{Res}_{E/\mathbb{F}_p}$  is the residue map for the  $(N-1)$  dimensional local field  $E$  defined by Kato [Ka1, §2].

**2.2. Global ramification theory.** Let  $X$  be a normal variety over a perfect field  $k$ . Let  $U \subset X$  be an open subscheme which is smooth over  $k$  and whose reduced complement  $C \subset X$  is the support of an effective Cartier divisor. Our aim in this section is to introduce the abelian fundamental group  $\pi_1^{\mathrm{ab}}(X, D)$  classifying abelian étale coverings of  $U$  with ramification bounded by  $D$ . Here  $D \in \mathrm{Div}(X)^+$  is an effective divisor with support in  $C$ .

Let  $I$  be the set of generic points of  $C$  and  $C_\lambda = \overline{\{\lambda\}}$  for  $\lambda \in I$ . For  $\lambda \in I$  let  $K_\lambda$  be the henselization of  $K = k(X)$  at  $\lambda$ . Note that  $K_\lambda$  is a henselian discrete valuation field with residue field  $k(C_\lambda)$ . We write  $H^1(U)$  for the étale cohomology group  $H^1(U, \mathbb{Q}/\mathbb{Z})$ .

**Proposition 2.6.**

- (1) Assume  $C$  is regular at a closed point  $x$  and  $x \in C_\lambda$  for  $\lambda \in I$ . Let  $F \in Z_1(X, C)^+$  be such that  $F \cap C$  at  $x$  and let  $k(F)_x$  be the henselization of  $k(F)$  at  $x$ . Take  $\chi \in H^1(U)$  and let  $\chi|_{K_\lambda} \in H^1(K_\lambda)$  and  $\chi|_{F,x} \in H^1(k(F)_x)$  be its restrictions. For an integer  $m \geq 0$ , we have an implication:

$$\chi|_{K_\lambda} \in \mathrm{fil}_m H^1(K_\lambda) \implies \chi|_{F,x} \in \mathrm{fil}_m H^1(k(F)_x).$$

- (2) Assume  $C = C_\lambda$  is regular and irreducible. Let  $T_X$  be the tangent sheaf of  $X$ . There is a dense open subset  $V_\chi \subset \mathbb{P}(T_X|_{C_\lambda})$  (depending on  $\chi$ ) such that for any integral  $F \in Z_1(X, C)^+$  and for any  $x$  with  $F \cap C$  at  $x$  the implication

$$T_F(x) \in V_\chi \implies \mathrm{ar}_{K_\lambda}(\chi|_{K_\lambda}) = \mathrm{ar}_{k(F)_x}(\chi|_{F,x})$$

holds.

- (3) Assume  $C$  is a simple normal crossing divisor in a neighborhood of  $x$ . Let  $g : X' = \mathrm{Bl}_x(X) \rightarrow X$  be the blowup at  $x$  and  $E \subset X'$  be the exceptional divisor and  $K_E$  be the henselization of  $K$  at its generic point. For a Cartier divisor  $D$  supported on  $C$  we put

$$m_E = \sum_{\lambda \in I_x} m_\lambda(D),$$

where  $I_x$  be the set of irreducible components of  $C$  containing  $x$  and  $m_\lambda(D)$  is the multiplicity of  $D$  at  $\lambda$ . Then, for

$$\chi \in \mathrm{Ker}(H^1(U) \rightarrow \bigoplus_{\lambda \in I_x} H^1(K_\lambda)/\mathrm{fil}_{m_\lambda(D)} H^1(K_\lambda)),$$

we have  $\chi|_{K_E} \in \mathrm{fil}_{m_E} H^1(K_E)$ .

*Proof.* (1) and (2) follow from [Ma, (7.2.1)]. (3) is proved by the same argument as [Ka2, Th.(8.1)] using [Ma, Cor.4.2.2] instead of [Ka2, Th.(7.1)].  $\square$

**Corollary 2.7.** *Assume  $C$  is a simple normal crossing divisor. For  $\chi \in H^1(U)$  and a Cartier divisor  $D$  supported on  $C$ , the following are equivalent*

- (1) *for all generic points  $\lambda$  of  $C$  we have  $\chi|_{K_\lambda} \in \text{fil}_{m_\lambda(D)} H^1(K_\lambda)$ ,*
- (2) *for all integral  $Z \in Z_1(X, C)^+$  and  $x \in Z_\infty$ , we have (see Definition 1.2)*

$$\chi|_{Z,x} \in \text{fil}_{m_x(\psi_Z^* D)} H^1(k(Z)_x) .$$

*Here  $\chi|_{Z,x} \in H^1(k(Z)_x)$  is the restriction of  $\chi$  and  $m_x$  is the multiplicity at  $x$ .*

*Proof.* The corollary follows from the proposition by observing that for integral  $Z \in Z_1(X, C)^+$  there is a chain of blowups in closed points such that the strict transform of  $Z$  becomes smooth and such that its intersection with the total transform of  $C$  is transversal.  $\square$

For general  $X$  and  $C$ , not necessarily of normal crossing, we make the following definition.

**Definition 2.8.** For  $D \in \text{Div}(X)^+$  with support in  $C$  we define  $\text{fil}_D H^1(U)$  to be the subgroup of  $\chi \in H^1(U)$  satisfying property (2) in Corollary 2.7. Define

$$(2.5) \quad \pi_1^{\text{ab}}(X, D) = \text{Hom}(\text{fil}_D H^1(U), \mathbb{Q}/\mathbb{Z}),$$

endowed with the usual pro-finite topology of the dual.

One should think of  $\pi_1^{\text{ab}}(X, D)$  as the quotient of  $\pi_1^{\text{ab}}(U)$  classifying abelian étale coverings of  $U$  with ramification bounded by  $D$ .

**Proposition 2.9.** *The filtration  $\text{fil}_D H^1(U)$  is exhaustive, i.e.*

$$\bigcup_D \text{fil}_D H^1(U) = H^1(U),$$

where  $D \in \text{Div}(X)^+$  runs through all divisors with support in  $C$ .

A proof can be found in [EK, Sec. 3.3].

### 3. EXISTENCE THEOREM

In this section  $k$  is assumed to be finite. Let  $U$  be a smooth variety over  $k$ . Choose a compactification  $U \subset X$  with  $X$  normal and proper over  $k$  such that  $X \setminus U$  is the support of an effective Cartier divisor on  $X$ . Put  $K = k(X)$ . In §1 we defined the relative Chow group of zero cycles  $C(X, D)$ , where  $D \in \text{Div}(X)^+$  is a Cartier divisor with support in  $C$ . We endow this relative Chow group with the discrete topology. We endow the group

$$C(U) = \varprojlim_D C(X, D)$$

with the inverse limit topology. Here  $D$  runs through all effective Cartier divisors on  $X$  with support in  $C$ .

**Lemma 3.1.** *The topological group  $C(U)$  does not depend on the choice of the compactification  $X$  of  $U$ .*

*Proof.* Let us write  $C(U \subset X)$  for the class group relative to the compactification  $X$  in the following. Assume  $U \subset X_1$  and  $U \subset X_2$  are two compactifications. Considering the normalization of the Zariski closure of the diagonal  $U \rightarrow X_1 \times_k X_2$ , we may assume that there is a morphism  $f : X_2 \rightarrow X_1$  which is the identity on  $U$ . It is then sufficient to show that the pushforward map (1.2)

$$(3.1) \quad f_* : C(U \subset X_2) \rightarrow C(U \subset X_1)$$

is an isomorphism. For an effective Cartier divisor  $D$  on  $X_1$  with support in  $X_1 \setminus U$ , one easily see that  $f_* : C(X_2, f^*D) \rightarrow C(X_1, D)$  is an isomorphism (see Definition 1.4(2)). As the divisors  $f^*D$  are cofinal in the system of all divisors on  $X_2$  with support in  $X_2 \setminus U$ , the isomorphy of (3.1) follows.  $\square$

In fact it is also clear from the proof that  $U \mapsto C(U)$  is a covariant functor from the category of smooth varieties over  $k$  to the category of topological abelian groups.

**Proposition 3.2.** *There is a unique continuous reciprocity homomorphism  $\rho_U$  making the diagram*

$$\begin{array}{ccc} Z_0(U) & \longrightarrow & C(U) \\ & \searrow & \downarrow \rho_U \\ & & \pi_1^{\text{ab}}(U) \end{array}$$

*commutative. Here the diagonal arrow is induced by the Frobenius homomorphisms  $\text{Frob}_x : \mathbb{Z} \rightarrow \pi_1^{\text{ab}}(U)$  for closed points  $x \in U$ . Moreover,  $\rho_U$  induces a homomorphism*

$$\rho_{X,D} : C(X, D) \rightarrow \pi_1^{\text{ab}}(X, D).$$

Recall that the pro-finite fundamental group  $\pi_1^{\text{ab}}(X, D)$  classifies abelian étale coverings of  $U$  with ramification over  $C$  bounded by the divisor  $D$ , see Definition 2.8. In what follows write  $M^\vee = \text{Hom}_{\text{cont}}(M, \mathbb{Q}/\mathbb{Z})$  for a topological abelian group  $M$ , where we endow  $\mathbb{Q}/\mathbb{Z}$  with the discrete topology.

*Proof of Proposition 3.2.* In [Wi], [KeSc] a continuous reciprocity homomorphism  $r_U : W(U) \rightarrow \pi_1^{\text{ab}}(U)$  is constructed. In order to accomplish the proof of the proposition we need some ramification theory. It is sufficient to show that for any character

$$\chi \in (\pi_1^{\text{ab}}(U))^\vee \cong H^1(U)$$

there is a divisor  $D \in \text{Div}(X)^+$  with support in  $C$  such that  $r_U^* \chi \in \text{Hom}(W(U) \rightarrow \mathbb{Q}/\mathbb{Z})$  factors through  $C(X, D)$ . In view of Definition 2.8, ramification properties of classical local class field theory (see [Se1, Sec. XV.2]) imply that the map  $r_U$  induces a map

$$\Psi_{X,D} : \text{fil}_D H^1(U) \longrightarrow C(X, D)^\vee.$$

Finally, the proposition follows from Proposition 2.9.  $\square$

Define topological groups  $C(U)^0$  and  $\pi_1^{\text{ab}}(U)^0$  as kernels in the commutative diagram

$$(3.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & C(U)^0 & \longrightarrow & C(U) & \xrightarrow{f_*} & C(\text{Spec} k) \\ & & \downarrow & & \downarrow \rho_U & & \downarrow \rho_k \\ 0 & \longrightarrow & \pi_1^{\text{ab}}(U)^0 & \longrightarrow & \pi_1^{\text{ab}}(U) & \xrightarrow{f_*} & \pi_1^{\text{ab}}(\text{Spec} k) \end{array}$$

where  $f : U \rightarrow \text{Spec} k$  is the natural morphism. Note that  $C(\text{Spec} k) = \mathbb{Z}$  and  $\rho_k$  maps  $1 \in \mathbb{Z}$  to the Frobenius over  $k$ . Let

$$\rho_U^0 : C(U)^0 \rightarrow \pi_1^{\text{ab}}(U)^0$$

be the induced map.

Our main theorem says:

**Theorem 3.3** (Existence Theorem). *Over a finite field  $k$  with  $\text{ch}(k) \neq 2$ ,  $\rho_U^0$  is an isomorphism of topological groups.*

**Corollary 3.4.** *Assume  $\text{ch}(k) \neq 2$ . For an effective divisor  $D \in \text{Div}(X)^+$  with support in  $C$ ,  $\rho_U^0$  induces an isomorphism of finite groups*

$$\rho_{X,D} : C(X, D)^0 \xrightarrow{\sim} \pi_1^{\text{ab}}(X, D)^0.$$

*Proof.* In view of Definition 2.8, the corollary follows from the Theorem 3.3 by using standard ramification properties in local class field theory as in explained in [Se1, Sec. XV.2].  $\square$

The proof of the Existence Theorem is begun in this section and completed in §5 assuming some technical lemmas that will be shown in later sections.

Now we start the proof of the Existence Theorem 3.3. Consider the following property

$\mathbf{I}_U$ :  $\rho_U$  induces a surjection

$$(3.3) \quad \Psi_U : H^1(U, \mathbb{Q}/\mathbb{Z}) \rightarrow C(U)^\vee = \text{Hom}_{\text{cont}}(C(U), \mathbb{Q}/\mathbb{Z}).$$

Note that we already know that the map (3.3) is injective by Chebotarev density theorem [Se2].

We now give an overview of the steps in the proof of the Existence Theorem 3.3.

- In Lemma 3.5 we show that property  $\mathbf{I}_U$  implies the Existence Theorem for the triple  $(X, C, U)$ .
- In Lemma 3.6 combined with de Jong's alteration theorem we show how to reduce the proof of  $\mathbf{I}_U$  to the situation where  $C$  is a simple normal crossing divisor.
- We use a Lefschetz hyperplane theorem [KeS] ( $C$  simple normal crossing) which allows us to reduce the proof of  $\mathbf{I}_U$  to the case  $\dim(X) = 2$ .
- In §4 we study (for  $\dim(X) = 2$ ) ramification filtrations on the Galois side and the class group side and compare graded pieces to complete the proof of  $\mathbf{I}_U$  in §5. The understanding of the filtration on the class group side is our key new ingredient.

**Lemma 3.5.** *Property  $\mathbf{I}_U$  implies that the map  $\rho_U$  in Theorem 3.3 is an isomorphism of topological groups.*

*Proof.* As  $\pi_1^{\text{ab}}(X, D)^0$  is finite by [KeS] for any effective divisor  $D$  with  $|D| \subset C$ , it is enough to show that  $\rho_U$  induces an isomorphism  $C(X, D)^0 \rightarrow \pi_1^{\text{ab}}(X, D)^0$  of abstract groups. It is sufficient to show that dually

$$\Psi_{X,D} : \text{fil}_D H^1(U) \rightarrow C(X, D)^\vee$$

is an isomorphism. The latter is a direct consequence of  $\mathbf{I}_U$  and classical ramification theory for local fields.  $\square$

We next introduce certain reduction techniques for property  $\mathbf{I}_U$ , based on methods of Wiesend. In the following lemma we denote by  $f : X' \rightarrow X$  an alteration with  $X'$  normal. By  $U' \subset f^{-1}(U)$  we denote an open smooth subscheme of  $X'$ , which is the complement of the support of an effective Cartier divisor. We use the notation

$$f : U' \rightarrow U, \quad C' = X' \setminus U'.$$

**Lemma 3.6** (Wiesend trick).

- (i) For  $f : X' \rightarrow X$  and  $U' \subset f^{-1}(U)$  as above, the implication  $\mathbf{I}_{U'} \Rightarrow \mathbf{I}_U$  holds.
- (ii) Assume that for any character  $\chi \in C(U)^\vee$  we can find  $f : X' \rightarrow X$  and  $U' \subset X'$  as above such that  $f^*(\chi) = 0$ . Then property  $\mathbf{I}_U$  holds.

*Proof.* We first explain the proof of (i). Consider the cartesian square of abstract groups

$$(3.4) \quad \begin{array}{ccc} H^1(U) & \xrightarrow{\Psi_U} & C(U)^\vee \\ f^* \downarrow & & \downarrow f^* \\ H^1(U') & \xrightarrow{\Psi_{U'}} & C(U')^\vee \end{array}$$

It is sufficient to see that a character  $\chi \in C(U)^\vee$  such that  $f^*(\chi)$  is of the form  $\Psi_{U'}(\sigma)$  with  $\sigma \in H^1(U')$  is in the image of  $\Psi_U$ . We can choose another alteration  $f' : X'' \rightarrow X'$  with the property that  $f'^{-1}(\sigma) = 0$ . This means that without loss of generality we can assume that  $f^*(\chi) = 0 \in C(U')$ .

Shrinking  $U'$  we can also assume that  $U' \rightarrow f(U') \subset X$  is the composition of a finite surjective radicial map  $U' \rightarrow U_{\text{ét}}$  and a finite étale map  $U_{\text{ét}} \rightarrow f(U')$ . As the maps

$$H^1(U_{\text{ét}}) \rightarrow H^1(U') \quad \text{and} \quad C(U_{\text{ét}})^\vee \rightarrow C(U')^\vee$$

are isomorphisms we can without loss of generality assume that  $X' \rightarrow X$  is generically étale. In this situation we finally conclude that  $\chi$  is in the image of  $\Psi_U$  by using Wiesend's method, see [KeSc, Prop. 3.7].

The proof of (ii) is a variant of the proof of (i). □

**Lemma 3.7.** *Assume that property  $\mathbf{I}_U$  holds for all smooth varieties  $U$  with  $\dim(U) = 2$ . Then it holds for arbitrary smooth  $U$ .*

*Proof.* By Lemma 3.5 we obtain Corollary 3.4 for two-dimensional  $X$ . In the general case we reduce the proof of property  $\mathbf{I}_U$  to the case  $C$  is simple normal crossing and  $X$  is projective by Lemma 3.6 and de Jong's alteration theorem [dJ]. This means that for such  $(X, C, U)$  we have to show that the map

$$C(X, D)^0 \rightarrow \pi_1^{\text{ab}}(X, D)^0$$

is an isomorphism for all  $D$ .

Let  $\mathcal{L}$  be an ample line bundle on  $X$ . Let  $i : Y \hookrightarrow X$  be a smooth hypersurface section, which is the zero locus of some section of  $\mathcal{L}^{\otimes n}$  ( $n \gg 0$ ), such that  $Y \times_X C$  is a reduced simple normal crossing divisor on  $Y$  and let  $E = Y \times_X D$ . Consider the commutative diagram

$$\begin{array}{ccccc} C(Y, E)^0 & \xrightarrow[\rho_{Y,E}]{\sim} & \pi_1^{\text{ab}}(Y, E)^0 & & \\ i_* \downarrow & & \downarrow \wr & & \\ Z_0(U)^0 & \longrightarrow & C(X, D)^0 & \xrightarrow[\rho_{X,D}]{\twoheadrightarrow} & \pi_1^{\text{ab}}(X, D)^0 \end{array}$$

The map  $\rho_{Y,E}$  is an isomorphism by induction on dimension. The right vertical map is an isomorphism for  $n$  sufficiently large [KeS]. The map  $\rho_{X,D}$  is surjective because of Chebotarev density [Se2] and the finiteness of  $\pi_1^{\text{ab}}(X, D)^0$ , see [KeS]. So we have to show injectivity of  $\rho_{X,D}$ .

For an  $\alpha \in C(X, D)^0$  with  $\rho_{X,D}(\alpha) = 0$  use a Bertini argument to choose  $Y$  as above which contains the support of a lift of  $\alpha$  to  $Z_0(U)$ . Then  $\alpha$  is in the image of  $i_*$ . A diagram chase shows that  $\alpha = 0$ . □

## 4. CYCLE CONDUCTOR

Let the notation be as in §1. Let  $X = (X, C)$  be in  $\mathcal{C}$  (cf. Definition 1.9). Let  $\{C_\lambda\}_{\lambda \in I}$  be the set of prime components of  $C$ . Fix a Cartier divisor

$$(4.1) \quad D = \sum_{\lambda \in I} m_\lambda C_\lambda \quad \text{with } m_\lambda \geq 2.$$

For a Cartier divisor  $F$  on  $X$  and  $Z \in \text{Div}(X, C)^+$ , we write

$$\mathcal{O}_Z(F) = \mathcal{O}_Z \otimes_{\mathcal{O}_X} \mathcal{O}_X(F).$$

For the moment we assume  $k$  is finite. In this section we will construct the key homomorphism

$$(4.2) \quad \text{cc}_{X,D} : C(X, D)^\vee \rightarrow H^0(C, \Omega_X^1(D + \Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C),$$

and state its basic properties. Here  $\Xi \in \text{Div}(X, C)^+$  is some sufficiently big Cartier divisor introduced below. First we note the canonical duality isomorphism

$$(4.3) \quad H^0(C, \Omega_X^1(D + \Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C) \simeq H^1(C, \Omega_X^1(-D - C - \Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C)^\vee.$$

Indeed, let  $\omega_C$  be the dualizing sheaf of  $C$ . We have

$$\omega_C \simeq \mathcal{E}xt_{\mathcal{O}_X}^1(\mathcal{O}_C, \Omega_X^2) \simeq \Omega_X^2(C) \otimes_{\mathcal{O}_X} \mathcal{O}_C,$$

where the second isomorphism follows from the long exact sequence for  $\mathcal{E}xt$  induced by the exact sequence  $0 \rightarrow \mathcal{O}_X(-C) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_C \rightarrow 0$ . Thus the Serre duality implies that the pairing

$$\Omega_X^1(D + \Xi) \otimes_{\mathcal{O}_X} \Omega_X^1(-D + C - \Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C \rightarrow \Omega_X^2(C) \otimes_{\mathcal{O}_X} \mathcal{O}_C \simeq \omega_C$$

induces a perfect pairing of abelian groups

$$(4.4) \quad H^0(C, \Omega_X^1(D + \Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C) \times H^1(C, \Omega_X^1(-D + C - \Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C) \rightarrow H^1(C, \omega_C) \xrightarrow{\text{Tr}_{C/\mathbb{F}_p}} \mathbb{Z}/p\mathbb{Z},$$

where  $\text{Tr}_{C/\mathbb{F}_p}$  is the composite  $H^1(C, \omega_C) \xrightarrow{\text{Tr}_{C/k}} k \xrightarrow{\text{Tr}_{k/\mathbb{F}_p}} \mathbb{Z}/p\mathbb{Z}$ . This induces (4.3). Hence the construction of (4.2) is reduced to that of its dual map:

$$H^1(C, \Omega_X^1(-D + C - \Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C) \rightarrow C(X, D).$$

or equivalently that of a map (see Theorem 4.5 below):

$$(4.5) \quad \phi_{X,D} : H^1(C, \Omega_X^1(-D - \Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C) \rightarrow W(U)/\widehat{F}^{(D+C)}W(X, C)$$

for a Cartier divisor

$$(4.6) \quad D = \sum_{\lambda \in I} m_\lambda C_\lambda \quad \text{with } m_\lambda \geq 1.$$

In fact this map turns out to exist over a general perfect field  $k$  at least replacing  $W(U)/\widehat{F}^{(D+C)}W(X, C)$  by its partial  $p$ -adic completion: We will get a natural map

$$\phi_{X,D} : H^1(C, \Omega_X^1(-D - \Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C) \rightarrow \lim_{\leftarrow n} (W(U)/\widehat{F}^{(D+C)}W(X, C) + p^n \widehat{F}^{(1)}W(U))$$

(see Theorem 4.5 and Remark 4.6). We call  $\text{cc}_{X,D}$  the *cycle conductor* for  $(X, D)$ .

In what follows  $k$  is only assumed to be perfect. For a regular closed point  $x$  of  $C$ , there is a natural isomorphism

$$H_x^1(C, \Omega_X^1(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_C) \simeq \frac{\Omega_X^1(-D) \otimes_{\mathcal{O}_X} k(C_\lambda)}{\Omega_X^1(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C,x}}$$

where  $C_\lambda$  is the irreducible component of  $C$  containing  $x$ . Thus we have a natural map

$$(4.7) \quad \iota_x : \frac{\Omega_X^1(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C,x}(x)}{\Omega_X^1(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C,x}} \rightarrow H_x^1(C, \Omega_X^1(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_C) \rightarrow H^1(C, \Omega_X^1(-D-\Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C).$$

We start with the construction of the local component  $\phi_{X,D} \circ \iota_x$  of  $\phi_{X,D}$  for  $x \in C \setminus \Xi$ .

Fix a regular closed point  $x$  of  $C$ . For  $g \in \mathcal{O}_{X,x}$ , let  $\text{div}_{X,x}(g)$  denote the effective Cartier divisor on  $X$  obtained from  $\text{div}_X(g)$  by removing its components which do not contain  $x$ . Note  $\text{div}_{X,x}(g) = \text{div}_{X,x}(ug)$  for  $u \in \mathcal{O}_{X,x}^\times$ . Put

$$(4.8) \quad \Lambda_x = \frac{\Omega_X^1(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C,x}(x)}{\Omega_X^1(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C,x}}.$$

Take a system of regular parameter  $(\pi, f)$  of  $\mathcal{O}_{X,x}$  such that  $\pi$  is a local parameter of  $C$  at  $x$ . We have

$$\Omega_{X,x}^1 = \mathcal{O}_{X,x} \cdot d\pi \oplus \mathcal{O}_{X,x} \cdot df.$$

Hence any element  $\xi \in \Lambda_x$  is written in a unique way as

$$\xi = \frac{1}{f}(\alpha d\pi + \beta df) \quad \text{with } \alpha, \beta \in \mathcal{O}_{X,x}(-D) \bmod \mathcal{O}_{X,x}(-D-C).$$

We define a map

$$\mu_{\pi,f} : \Lambda_x \rightarrow W(U)/F^{(D+C)}W(X, C)$$

by

$$(4.9) \quad \mu_{\pi,f}\left(\frac{1}{f}(\alpha d\pi + \beta df)\right) = \{1 + (\beta - \alpha)\}_{F,x} + \{1 + \alpha\}_{F_\pi,x},$$

where  $F = \text{div}_{X,x}(f) \in \text{Div}(X, C)^+$  and  $F_\pi = \text{div}_{X,x}(f + \pi) \in \text{Div}(X, C)^+$  (see Definition 1.8).

**Proposition 4.1.** *The map  $\mu_{\pi,f}$  is independent of the choice of  $(\pi, f)$ .*

We use the following lemma (see Lemma 6.1).

**Lemma 4.2.** *Let  $x$  be a regular closed point of  $C$ . Let  $F, Z_1, Z_2 \in \text{Div}(X, C)^+$  be such that  $F \pitchfork C$  at  $x$ , and  $F \pitchfork Z_i$  and  $Z_i \pitchfork C$  at  $x$  for  $i = 1, 2$ . Let  $(\pi, f)$  be a system of regular parameters such that*

$$F = \text{div}_{X,x}(f), \quad Z_1 = \text{div}_{X,x}(u_1 f + \pi), \quad Z_2 = \text{div}_{X,x}(u_2 f + \pi), \quad C = \text{div}_{X,x}(\pi),$$

where  $u_1, u_2 \in \mathcal{O}_{X,x}^\times$ . For  $\alpha \in \mathcal{O}_{X,x}(-D)$ , we have

$$\{1 - (u_1 - u_2)\alpha\}_{F,x} + \{1 - u_1\alpha\}_{Z_1,x} - \{1 - u_2\alpha\}_{Z_2,x} \in F^{(D+C)}W(X, C).$$

Proposition 4.1 follows from the following claims.

**Claim 4.3.** *For a system of regular parameters  $(\pi', f')$  where  $f' = uf$  and  $\pi' = v\pi$  with  $u, v \in \mathcal{O}_{X,x}^\times$ , we have  $\mu_{\pi,f} = \mu_{\pi',f'}$ .*

**Claim 4.4.** *For a system of regular parameters  $(\pi, f')$  where  $f' = uf + \pi$  with  $u \in \mathcal{O}_{X,x}^\times$ , we have  $\mu_{\pi,f} = \mu_{\pi,f'}$ .*

*Proof of Claim 4.3* Put

$$F'_{\pi'} = \text{div}_{X,x}(f' + \pi') = \text{div}_{X,x}(uv^{-1}f + \pi).$$

For  $\alpha, \beta \in \mathcal{O}_{C,x}(-D)$ , we compute in  $\Lambda_x$

$$(4.10) \quad \begin{aligned} \xi &:= \frac{1}{f}(\alpha d\pi + \beta df) = \frac{u}{f'}(\alpha d(v^{-1}\pi') + \beta d(u^{-1}f')) \\ &= \frac{1}{f'}(\alpha uv^{-1}d\pi' + \beta df') + \beta udu^{-1} = \frac{1}{f'}(\alpha uv^{-1}d\pi' + \beta df') \end{aligned}$$

Hence

$$\mu_{(\pi',f')}(\xi) = \{1 + (\beta - \alpha uv^{-1})\}_{F,x} + \{1 + \alpha uv^{-1}\}_{F'_{\pi'},x} \in W(U)/F^{(D+C)}W(X, C),$$

$$\mu_{(\pi,f)}(\xi) - \mu_{(\pi',f')}(\xi) = \{1 - \alpha(1 - uv^{-1})\}_{F,x} + \{1 + \alpha\}_{F_{\pi},x} - \{1 + \alpha uv^{-1}\}_{F'_{\pi'},x},$$

which vanishes by Lemma 4.2 applied to

$$F = \operatorname{div}_{X,x}(f), \quad Z_1 = F_{\pi} = \operatorname{div}_{X,x}(f + \pi), \quad Z_2 = F'_{\pi'} = \operatorname{div}_{X,x}(uv^{-1}f + \pi).$$

This completes the proof of Claim 4.3.

*Proof of Claim 4.4.* Put

$$F' = \operatorname{div}_{X,x}(f') = \operatorname{div}_{X,x}(uf + \pi), \quad F'_{\pi} = \operatorname{div}_{X,x}(f' + \pi) = \operatorname{div}_{X,x}\left(\frac{u}{2}f + \pi\right).$$

Here we used the assumption  $p \neq 2$ . For  $\alpha, \beta \in \mathcal{O}_{C,x}(-D)$ , we compute in  $\Lambda_x$

$$(4.11) \quad \begin{aligned} \xi &:= \frac{1}{f}(\alpha d\pi + \beta df) = \frac{u}{f'}(\alpha d\pi + \beta u^{-1}(df' - d\pi)) + \beta udu^{-1} \\ &= \frac{1}{f'}((\alpha u - \beta)d\pi + \beta df') \end{aligned}$$

Hence

$$\mu_{(\pi,f')}(\xi) = \{1 + (2\beta - \alpha u)\}_{F',x} + \{1 + (\alpha u - \beta)\}_{F'_{\pi},x} \in W(U)/F^{(D+C)}W(X, C),$$

$$(4.12) \quad \begin{aligned} \mu_{(\pi,f)}(\xi) - \mu_{(\pi,f')}(\xi) &= \\ &= \{1 + (\beta - \alpha)\}_{F,x} + \{1 + \alpha\}_{F_{\pi},x} - \{1 + (2\beta - \alpha u)\}_{F',x} - \{1 + (\alpha u - \beta)\}_{F'_{\pi},x}. \end{aligned}$$

By Lemma 4.2 applied to

$$F = \operatorname{div}_{X,x}(f), \quad Z_1 = F_{\pi} = \operatorname{div}_{X,x}(f + \pi), \quad Z_2 = F' = \operatorname{div}_{X,x}(uf + \pi),$$

we get

$$(4.13) \quad \{1 - \alpha(1 - u)\}_{F,x} + \{1 + \alpha\}_{F_{\pi},x} - \{1 + \alpha u\}_{F',x} = 0 \in W(U)/F^{(D+C)}W(X, C).$$

By Lemma 4.2 applied to

$$F = \operatorname{div}_{X,x}(uf), \quad Z_1 = F' = \operatorname{div}_{X,x}(uf + \pi), \quad Z_2 = F'_{\pi} = \operatorname{div}_{X,x}\left(\frac{1}{2}uf + \pi\right),$$

we get

$$(4.14) \quad \{1 + (\alpha u - \beta)\}_{F,x} + \{1 + 2(\alpha u - \beta)\}_{F',x} - \{1 + (\alpha u - \beta)\}_{F'_{\pi},x} = 0 \in W(U)/F^{(D+C)}W(X, C).$$

Summing up (4.13) and (4.14), we get  $\mu_{(\pi,f)}(\xi) - \mu_{(\pi,f')}(\xi) = 0$  by (4.12). This completes the proof of Claim 4.3 and hence that of Proposition 4.1.  $\square$

For a regular closed point  $x$  of  $C$ , let

$$(4.15) \quad \mu_x : \frac{\Omega_X^1(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C,x}(x)}{\Omega_X^1(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C,x}} \rightarrow W(U)/F^{(D+C)}W(X, C)$$

be the map in Proposition 4.1. We now state the key theorems for the proof of Theorem 3.3. The proof of Theorems 4.5 and 4.7 will be given in §7 and §8 respectively.

**Theorem 4.5.** *There exists an effective Cartier divisor  $\Xi$  independent of  $D$  as in (4.6) such that  $C_{\text{sing}} \subset \Xi$  and a natural map*

$$\phi_{X,D} : H^1(C, \Omega_X^1(-D-\Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C) \rightarrow \varprojlim_n (W(U)/\widehat{F}^{(D+C)}W(X, C) + p^n \widehat{F}^{(1)}W(U))$$

for which the following conditions hold:

(i) For any closed point  $x$  of  $C \setminus \Xi$ , the following diagram is commutative:

$$(4.16) \quad \begin{array}{ccc} & H^1(C, \Omega_X^1(-D-\Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C) & \\ \nearrow \iota_x & & \downarrow \phi_{X,D} \\ \frac{\Omega_X^1(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C,x}(x)}{\Omega_X^1(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C,x}} & & \\ \searrow \mu_x & & \downarrow \\ & \varprojlim_n (W(U)/\widehat{F}^{(D+C)}W(X, C) + p^n \widehat{F}^{(1)}W(U)). & \end{array}$$

(ii)  $\text{Image}(\phi_{X,D}) = \text{Image}(\widehat{F}^{(D)}W(X, C))$ .

*Remark 4.6.*  $\phi_{X,D}$  lifts to a natural map

$$\phi_{X,D} : H^1(C, \Omega_X^1(-D-\Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C) \rightarrow W(U)/\widehat{F}^{(D+C)}W(X, C)$$

under one of the following conditions:

- (i)  $D \geq 2C$ ,
- (ii)  $k$  is finite.

The case (ii) follows from Corollary 5.2. As for the case (i), see Remark 7.7.

**Theorem 4.7.** *Assume  $k$  is finite. For  $D$  as (4.1) let*

$$\text{cc}_{X,D} : C(X, D)^\vee \rightarrow H^0(C, \Omega_X^1(D+\Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C),$$

be induced by  $\phi_{X,D-C}$  by (4.3). For  $\lambda \in I$ , let  $m_\lambda$  be as in (4.1) and

$$\text{rar}_{K_\lambda} : \text{gr}_{m_\lambda} H^1(K_\lambda) \rightarrow \text{fil}_{m_\lambda} \Omega_{K_\lambda}^1$$

be the refined Swan for  $K_\lambda$  in Theorem 2.3. We note

$$\Omega_X^1(D) \otimes_{\mathcal{O}_X} k(C_\lambda) \simeq \text{gr}_{m_\lambda} \Omega_{K_\lambda}^1.$$

Then the following diagram commutes

$$\begin{array}{ccccc} \text{fil}_D H^1(U) & \longrightarrow & \text{fil}_{m_\lambda} H^1(K_\lambda) & \xrightarrow{\text{rar}_{K_\lambda}} & \Omega_X^1(D) \otimes_{\mathcal{O}_X} k(C_\lambda) \\ \downarrow \Psi_{X,D} & & & & \uparrow \iota_\lambda \\ C(X, D)^\vee & \xrightarrow{\text{cc}_{X,D}} & H^0(C, \Omega_X^1(D+\Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C) & & \end{array}$$

## 5. PROOF OF EXISTENCE THEOREM

Let the notation be as in §3. In this section we always assume  $\text{ch}(k) \neq 2$  and  $\dim(X) = 2$ , but we do not assume that  $C$  is simple normal crossing or that  $X$  is smooth. We prove property  $\mathbf{I}_U$  in this case, which finishes the proof of the Existence Theorem 3.3 by Lemma 3.7. We first make some reduction steps.

We start with the tame case, which is essentially due to Wiesend. Note that the second statement of the following proposition is motivated by the fact that an abelian étale covering  $U' \rightarrow U$  is tame over  $C$  if and only if its pullback to integral  $F \in Z_1(X, C)^+$  such that  $F \pitchfork C$  is tame over  $C \cap F$ , see Proposition 2.6.

**Proposition 5.1.** *Assume  $\dim(X) = 2$ . The reciprocity map  $\rho_U$  induces an isomorphism of finite groups*

$$C(X, C)^0 \xrightarrow{\sim} \pi_1^{\text{ab}}(X, C)^0.$$

Moreover the closure of the image of  $F_{\pitchfork}^{(C)}W(X, C)$  in  $C(U)$  is equal to  $\widehat{F}^{(C)}W(X, C)$ .

*Proof.* It is shown in [KeSc, Thm. 8.3] (see also Remark 1.5) that  $\rho_U$  induces an isomorphism

$$W(U)^0 / \widehat{F}^{(1)}W(U) \xrightarrow{\sim} \pi_1^{\text{ab}}(X, C)^0.$$

The verbatim same argument shows the proposition.  $\square$

**Corollary 5.2.** *Let the assumptions be as in Proposition 5.1. For any Cartier divisor on  $X$  with  $|D| \subset C$ ,  $C(X, D)^0$  is torsion of finite exponent and  $\widehat{F}^{(1)}W(U) / \widehat{F}^{(D)}W(U)$  is of finite exponent of  $p$ -power.*

*Proof.* Without loss of generality we assume  $|D| = C$ . By Proposition 5.1, we have

$$W(U)^0 / (F_{\pitchfork}^{(C)}W(X, C) + \widehat{F}^{(D)}W(U)) \cong W(U)^0 / \widehat{F}^{(C)}W(U),$$

and it is isomorphic to  $\pi_1^{\text{ab}}(X, C)^0$ , which is finite by [KeSc, Th.2.7]. On the other hand  $p^m F_{\pitchfork}^{(C)}W(X, C) \subset F^{(D)}W(X, C)$  if  $p^m C \geq D$  since for  $F \in \text{Div}(X, C)^+$  such that  $F \pitchfork C$  at  $x \in F \cap C$ ,  $(1 + \mathcal{O}_{F,x}(-C))^{p^m} \subset 1 + \mathcal{O}_{F,x}(-p^m C)$ .  $\square$

Now we turn to the proof of property  $\mathbf{I}_U$  in the wild case. By Wiesend's trick, Lemma 3.6, and a standard fibration technique [SGA4, XI, Prop. 3.3] we can assume that there is a proper smooth curve  $S$  over  $k$  and morphisms

$$f : X \rightarrow S \quad \text{and} \quad \sigma : S \rightarrow X$$

where  $f$  is a proper surjective morphism with smooth generic fiber and  $\sigma$  is a section of  $f$ . Let  $I$  be the set of generic points  $\lambda$  of  $C$  which lie over the generic point  $\eta$  of  $S$ . We can assume:

- $f(\sigma(S) \cap C)$  does not contain  $\eta$ .
- The induced morphism  $C \rightarrow S$  induces an isomorphism on each irreducible component  $C_\lambda$  for  $\lambda \in I$ . Here  $C_\lambda = \overline{\{\lambda\}}$ .
- $f|_U : U \rightarrow S$  is smooth.

Let us fix an algebraic closure  $\overline{k(S)}$  of  $k(S)$ . Write  $\bar{\eta} = \text{Spec } \overline{k(S)}$ . Let us consider pairs  $\Sigma = (T, \theta)$  where

- $T$  is the normalization of  $S$  in a finite subextension of  $k(S)$  in the field extension  $k(S) \subset \overline{k(S)}$
- $\theta$  is an effective divisor on  $T$ .

Clearly for such a  $\Sigma = (T, \theta)$  there is a canonical map  $T \rightarrow S$ . We define a directed partial ordering on the set of all  $\Sigma$  by setting

$$\Sigma_1 = (T_1, \theta_1) \leq \Sigma_2 = (T_2, \theta_2),$$

if  $k(T_1) \subset k(T_2)$ , which means that the map  $T_2 \rightarrow S$  factors canonically through

$$T_2 \xrightarrow{g_{\Sigma_2, \Sigma_1}} T_1 \rightarrow S$$

and if

$$g_{\Sigma_2, \Sigma_1}^*(\theta_1) \leq \theta_2.$$

By abuse of notation we also write  $\theta \in \text{Div}^+(X_\Sigma)$  for the pullback of  $\theta$  to

$$X_\Sigma = \text{normalization of } X \times_S T.$$

By  $U_\Sigma$  we denote the preimage of  $U$  in  $X_\Sigma \setminus \text{supp}(\theta)$ .

Using the compatibility of étale cohomology with directed inverse limits of schemes we get an isomorphism

$$(5.1) \quad \varinjlim_{\Sigma} H^1(U_\Sigma) \xrightarrow{\sim} H^1(U_{\bar{\eta}}).$$

Thinking of  $U_{\bar{\eta}}$  as a smooth curve over  $\bar{\eta}$  with compactification  $X_{\bar{\eta}}$  we endow the cohomology group  $H^1(U_{\bar{\eta}})$  with the ramification filtration

$$\text{fil}_m H^1(U_{\bar{\eta}}) = \text{Ker}(H^1(U) \rightarrow \bigoplus_{\lambda \in I} H^1(K_\lambda^-) / \text{fil}_m H^1(K_\lambda^-))$$

where  $K_\lambda^-$  is the henselization of  $X_{\bar{\eta}}$  at the preimage of  $\lambda$ . Combining (5.1) with Corollary 2.7 we get an isomorphism

$$(5.2) \quad \varinjlim_{\Sigma} \text{fil}_{mC+\theta} H^1(U_\Sigma) \xrightarrow{\sim} \text{fil}_m H^1(U_{\bar{\eta}}).$$

Here we write  $C \in \text{Div}^+(X_\Sigma)$  also for the pullback of  $C \in \text{Div}^+(X)$  to the scheme  $X_\Sigma$ . Note that the filtration on the left side is constructed in terms of curves on  $X_\Sigma$  as in Definition 2.8.

Composing (5.2) with the dual reciprocity map, see Proposition 3.2, we get a homomorphism

$$(5.3) \quad \Psi_{\bar{\eta}}^{(m)} : \text{fil}_m H^1(U_{\bar{\eta}}) \rightarrow \varinjlim_{\Sigma} C(X_\Sigma, mC + \theta)^\vee.$$

Wiesend's trick, see Lemma 3.6, tells us that the surjectivity of  $\Psi_{\bar{\eta}}^{(m)}$  implies property  $\mathbf{I}_U$ .

We next recall an idea of Wiesend, see [KeSc], which shows that  $\Psi_{\bar{\eta}}^{(1)}$  is surjective.

**Lemma 5.3** (Wiesend). *The map  $\Psi_{\bar{\eta}}^{(1)}$  is surjective.*

*Sketch of proof.* Consider  $\chi \in C(X_\Sigma, C + \theta)^\vee$  for some  $\Sigma = (T, \theta)$ . By Wiesend's trick Lemma 3.6 it is enough to construct a quasi-finite map  $U' \rightarrow U_\Sigma$  with dense image such that the pullback of  $\chi$  to  $U'$  vanishes. By Corollary 5.2 the map

$$\varinjlim_n \text{Hom}_{\text{cont}}(C(X_\Sigma, C + \theta), \mathbb{Z}/n) \xrightarrow{\sim} C(X_\Sigma, C + \theta)^\vee$$

is an isomorphism. This means that we can find  $n$  such that

$$\chi \in \text{Hom}_{\text{cont}}(C(X_\Sigma, C + \theta), \mathbb{Z}/n) \subset C(X_\Sigma, C + \theta)^\vee.$$

If we pull back  $\chi$  along the section  $\sigma : T \rightarrow X_\Sigma$  we get a character in  $C(T, \theta')^\vee$  for some divisor  $\theta'$ . By one-dimensional global class field theory it comes from a cohomology element in  $H^1(T \setminus |\theta'|)$  via the dual reciprocity map. Enlarging  $\Sigma$ , i.e. by making a base change in the base  $T$ , we can assume that this cohomology element vanishes.

The maximal abelian étale extension  $U'$  of  $(U_\Sigma)_\eta$  which splits over the image of  $\sigma$ , whose Galois group is  $n$ -torsion and which is tame over  $(X_\Sigma)_\eta$  is finite. Finally, use base change for tame fundamental groups and one-dimensional class field theory along the fibers of  $U_\Sigma \rightarrow T$  to deduce that the pullback of  $\chi$  to the class group of  $U'$ , or rather some spreading of  $U'$  to a scheme of finite type over  $T$ , vanishes.  $\square$

We now prove  $\Psi_\eta^{(m)}$  for  $m \geq 2$  by induction on  $m$ . Consider the exact localization sequence

$$0 \rightarrow H^1(X_{\bar{\eta}}) \rightarrow H^1(U_{\bar{\eta}}) \rightarrow \bigoplus_{\lambda \in I} H^1(K_\lambda^-) \xrightarrow{\iota} H^2(X_{\bar{\eta}}, \mathbb{Q}/\mathbb{Z}).$$

It induces the exact sequences in the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(X_{\bar{\eta}}) & \longrightarrow & \text{fil}_{m-1} H^1(U_{\bar{\eta}}) & \longrightarrow & \bigoplus_{\lambda \in I} \text{fil}_{m-1} H^1(K_\lambda^-) \xrightarrow{\iota} \iota(\text{fil}_{m-1}) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1(X_{\bar{\eta}}) & \longrightarrow & \text{fil}_m H^1(U_{\bar{\eta}}) & \longrightarrow & \bigoplus_{\lambda \in I} \text{fil}_m H^1(K_\lambda^-) \xrightarrow{\iota} \iota(\text{fil}_m) \longrightarrow 0 \end{array}$$

Here the vertical maps are the canonical inclusions. Taking cokernels of the vertical maps we get an exact sequence

$$(5.4) \quad 0 \rightarrow \text{gr}_m H^1(U_{\bar{\eta}}) \rightarrow \bigoplus_{\lambda \in I} \text{gr}_m H^1(K_\lambda^-) \xrightarrow{\iota} \iota(\text{fil}_m)/\iota(\text{fil}_{m-1}) \rightarrow 0.$$

The map  $\iota$  in (5.4) vanishes, because  $\iota(\text{fil}_m)/\iota(\text{fil}_{m-1})$  is a subquotient of the cohomology group  $H^2(X_{\bar{\eta}}, \mathbb{Q}/\mathbb{Z})$ , which has no  $p$ -torsion by [SGA4, X, Thm. 5.1], and  $\text{gr}_m H^1(K_\lambda^-)$  is a  $p$ -primary torsion group.

So we get a commutative diagram with exact columns

$$\begin{array}{ccc} \text{fil}_{m-1} H^1(U_{\bar{\eta}}) & \xrightarrow{\Psi_\eta^{(m-1)}} & \varinjlim_\Sigma C(X_\Sigma, (m-1)C + \theta)^\vee \\ \downarrow & & \downarrow \\ \text{fil}_m H^1(U_{\bar{\eta}}) & \xrightarrow{\Psi_\eta^{(m)}} & \varinjlim_\Sigma C(X_\Sigma, mC + \theta)^\vee \\ \downarrow & & \downarrow \text{cc} \\ \bigoplus_{\lambda \in I} \text{gr}_m H^1(K_\lambda^-) & \xrightarrow{\oplus_\lambda \text{rar}_\lambda} & \bigoplus_{\lambda \in I} \Omega_{X_{\bar{\eta}}}^1(mC) \otimes k(C_\lambda) \\ \downarrow & & \\ 0 & & \end{array}$$

The map  $\text{cc}$  is induced by the cycle conductor defined in §4 for a desingularization of  $X_\Sigma$ . The exactness of the right vertical sequence is deduced from Theorem 4.5. The surjective map  $\text{rar}_\lambda$  is the refined Artin conductor for  $K_\lambda^-$  recalled in Theorem 2.3. The lower square commutes by Theorem 4.7.

A diagram chase shows that the surjectivity of  $\Psi_\eta^{(m-1)}$  implies the surjectivity of  $\Psi_\eta^{(m)}$ . This finishes the induction and therefore the proof of Theorem 3.3.

## 6. KEY LEMMAS

Let the notation be as in §1. Let  $(X, C)$  be in  $\mathcal{C}$  (see Definition 1.9) and  $\{C_\lambda\}_{\lambda \in I}$  be the set of prime components of  $C$ . Fix a Cartier divisor

$$(6.1) \quad D = \sum_{\lambda \in I} m_\lambda C_\lambda \quad \text{with } m_\lambda \geq 1.$$

In this section we state three key lemmas which will be used in the proof of the main results. The first key lemma is a relation among three symbols.

**Lemma 6.1** (three term relation). *Let  $x$  be a regular closed point of  $C$ . Let  $F, Z_1, Z_2 \in \text{Div}(X, C)^+$  be such that  $F \cap C$  at  $x$ , and  $F \cap Z_i$  and  $Z_i \cap C$  at  $x$  for  $i = 1, 2$ . Let  $(\pi, f)$  be a system of regular parameters such that locally at  $x$ ,*

$$F = \text{div}_X(f), \quad Z_1 = \text{div}_X(u_1 f + \pi), \quad Z_2 = \text{div}_X(u_2 f + \pi), \quad C = \text{div}_X(\pi),$$

where  $u_1, u_2 \in \mathcal{O}_{X,x}^\times$ . For  $\alpha \in \mathcal{O}_{X,x}(-D)$ , we have

$$\{1 - (u_1 - u_2)\alpha\}_{F,x} + \{1 - u_1\alpha\}_{Z_1,x} - \{1 - u_2\alpha\}_{Z_2,x} \in F^{(D+C)}W(X, C).$$

To state the second key lemma, we introduce a definition.

**Definition 6.2.** Let  $X = (X, C)$  be in  $\mathcal{C}$ . Let  $F \in \text{Div}(X, C)^+$  be a reduced effective Cartier divisor such that  $F \cap C$ . Fix  $\pi, \pi_D, f \in \mathcal{O}_{X, F \cap C}$  such that locally at  $F \cap C$

$$C = \text{div}_X(\pi), \quad D = \text{div}_X(\pi_D), \quad F = \text{div}_X(f).$$

By the assumption  $(\pi, f)$  is a system of regular parameters in  $\mathcal{O}_{X, F \cap C}$ . For an integer  $e > 0$ , let  $\mathcal{P}_{D,e}(F)_{(X,C)} = \mathcal{P}_{D,e}(F)$  denotes the set of  $a \in H^0(X, \mathcal{O}_X(-D + eF))$  such that

$$\text{div}_X(a) = D - eF + W,$$

where  $W \in \text{Div}(X, C)^+$  such that  $W \cap F \cap C = \emptyset$ . For  $a \in \mathcal{P}_{D,e}(F)$ , put

$$Z_a = \text{div}_X(1 + a) + eF.$$

It follows immediately from the definition that  $Z_a \in \text{Div}(X, C)^+$  and  $Z_a \cap C = F \cap C$ , and that locally at  $F \cap C$ ,

$$(6.2) \quad Z_a = \text{div}_X(f^e + \pi_D \cdot u) \quad \text{with } u \in \mathcal{O}_{X, F \cap C}^\times.$$

Note

$$(6.3) \quad 1 - f^{e+1}\mathcal{O}_{Z_a, C \cap Z_a} = 1 + \pi_D f \mathcal{O}_{Z_a, C \cap Z_a} \subset 1 + \mathcal{O}_{Z_a, C \cap Z_a}(-D).$$

**Lemma 6.3** (increasing order). *Let  $Z_a$  with  $a \in \mathcal{P}_{D,e}(F)$  be as above and take  $x \in Z_a \cap C$  and  $\lambda \in I$  such that  $x \in C_\lambda$ . Assume*

$$(*) \quad H^1(X, \mathcal{O}_X(-2D - C + (e-1)F)) = H^1(C, \mathcal{O}_C(-2D + F)) = 0,$$

(1) *Assuming  $D \geq 2C$  and  $e \geq m_\lambda$ , we have*

$$\{1 + f^{e+1}\mathcal{O}_{Z_a, F \cap C}\}_{Z_a, x} \subset \widehat{F}^{(D+C)}W(X, C).$$

(2) *Assuming  $p \neq 2$  and  $e \geq m_\lambda(p^n - 1)$  for  $n > 0$ , we have*

$$\{1 + f^{e+1}\mathcal{O}_{Z_a, F \cap C}\}_{Z_a, x} \subset \widehat{F}^{(D+C)}W(X, C) + p^n \widehat{F}^{(1)}W(U).$$

*Remark 6.4.* By Serre's vanishing theorem the condition  $(*)$  of Lemma 10.1 is satisfied if  $F \in \mathcal{L}(d)$  for  $d > 0$  sufficiently large (see Definition 7.2 below).

The last key lemma concerns moving elements of  $W(U)$  to symbols on curves transversal to  $C$ . Take any dense open subset  $V \subset X$  containing the generic points of  $C$  and recall Definition 1.4.

**Lemma 6.5** (moving). *For any positive integer  $n$ , we have*

$$\widehat{F}^{(D)}W(X, C) \subset F_{\mathbb{M}V}^{(D)}W(X, C) + \widehat{F}^{(D+C)}W(X, C) + p^n \widehat{F}^{(1)}W(U).$$

*Remark 6.6.* If  $D \geq 2C$ , we get (cf. Remark 13.8)

$$\widehat{F}^{(D)}W(X, C) \subset F_{\mathbb{M}V}^{(D)}W(X, C) + \widehat{F}^{(D+C)}W(X, C).$$

The logical structure of the proofs of the above key lemmas is as follows:

Lemma 6.1  $\Leftarrow$  Lemma 9.1

Lemma 6.3  $\Leftarrow$  Lemma 10.1

Lemma 10.1(1)  $\Leftarrow$  Lemma 10.2 + Lemma 10.6(1)

$\Downarrow$

Lemma 11.1(1)  $\Rightarrow$  Lemma 12.3  $\Rightarrow$  Lemma 12.1

$\Downarrow$

Lemma 10.6(2) + Lemma 10.2

$\Downarrow$

Lemma 10.1(2)  $\Rightarrow$  Lemma 11.1(2)

Lemma 6.5  $\Leftarrow$  Lemma 11.1(2) + Lemma 9.1

In §9 we prove Lemma 9.1 from which Lemma 6.1 is deduced. Lemma 6.3 is an immediate consequence of Lemma 10.1 which consists of two parts (1) and (2). In §10 we prove Lemma 10.2 and Lemma 10.6(1) from which Lemma 10.1(1) is deduced. In §11 we prove Lemma 11.1(1) using Lemma 10.1(1). In §12 we prove Lemma 12.3 using Lemma 11.1(1) and prove Lemma 12.1 using Lemma 12.3. Lemma 10.6(2) is a direct consequence of Lemma 12.1 and Lemma 10.1(2) follows from Lemma 10.6(2) and Lemma 10.2 as Lemma 10.1(1) is deduced from Lemma 10.6(1) and Lemma 10.2 in §10. Finally, in §12 we prove Lemma 6.5 using Lemma 11.1(2) and Lemma 9.1.

## 7. PROOF OF KEY THEOREM I

In this section we prove Theorem 4.5. First note that (ii) follows from (i) thanks to Lemma 6.5.

**Lemma 7.1.** *Let  $L/k$  be a finite Galois extension with  $G = \text{Gal}(L/k)$ . If Theorem 4.5 holds for  $X_L = X \otimes_k L$ , it holds for  $X$ .*

*Proof.* For a regular closed point  $x$  of  $C$ , we have

$$\begin{array}{ccc} \bigoplus_{y|x} \frac{\Omega_{X_L}^1(-D_L) \otimes_{\mathcal{O}_{X_L}} \mathcal{O}_{C_L, y}(y)}{\Omega_{X_L}^1(-D_L) \otimes_{\mathcal{O}_{X_L}} \mathcal{O}_{C_L, y}} & \xrightarrow{\mu_y} & W(U_L) / \widehat{F}^{(D_L+C_L)} W(X_L, C_L) \\ \downarrow \text{Tr}_{L/k} & & \downarrow N_{L/k} \\ \frac{\Omega_X^1(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C, x}(x)}{\Omega_X^1(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C, x}} & \xrightarrow{\mu_x} & W(U) / \widehat{F}^{(D+C)} W(X, C) \end{array}$$

where the  $N_{L/k}$  is the norm map for  $X_L \rightarrow X$  and  $y$  ranges over the points of  $C_L$  lying over  $x$ . We also have a commutative diagrams

$$\begin{array}{ccc} \bigoplus_{y|x} \frac{\Omega_{X_L}^1(-D_L) \otimes_{\mathcal{O}_{X_L}} \mathcal{O}_{C_L, y}(y)}{\Omega_{X_L}^1(-D_L) \otimes_{\mathcal{O}_{X_L}} \mathcal{O}_{C_L, y}} & \xrightarrow{\iota_y} & H^1(C_L, \Omega_{X_L}^1(-D_L - \Xi_L) \otimes_{\mathcal{O}_{X_L}} \mathcal{O}_{C_L}) \\ \downarrow \text{Tr}_{L/k} & & \downarrow \text{Tr}_{L/k} \\ \frac{\Omega_X^1(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C, x}(x)}{\Omega_X^1(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C, x}} & \xrightarrow{\iota_x} & H^1(C, \Omega_X^1(-D - \Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C), \end{array}$$

and the trace map  $\text{Tr}_{L/k}$  induces isomorphisms

$$\begin{aligned} H^1(C_L, \Omega_{X_L}^1(-D_L - \Xi_L) \otimes_{\mathcal{O}_{X_L}} \mathcal{O}_{C_L})_G &\xrightarrow{\cong} H^1(C, \Omega_X^1(-D - \Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C), \\ \left( \bigoplus_{y|x} \frac{\Omega_{X_L}^1(-D_L) \otimes_{\mathcal{O}_{X_L}} \mathcal{O}_{C_L, y}(y)}{\Omega_{X_L}^1(-D_L) \otimes_{\mathcal{O}_{X_L}} \mathcal{O}_{C_L, y}} \right)_G &\xrightarrow{\cong} \frac{\Omega_X^1(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C, x}(x)}{\Omega_X^1(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C, x}} \end{aligned}$$

where  $M_G$  denotes the coinvariants of a  $G$ -module  $M$ . The lemma follows from these.  $\square$

**Definition 7.2.** Let  $(X, C)$  be in  $\mathcal{C}$  (see Definition 1.9).

- (1) Let  $H \subset X$  be a hyperplane section. For an integer  $d > 0$  let  $\mathcal{L}(d) = |dH|$  be the linear system on  $X$  of hypersurface sections of degree  $d$ . For  $t \in \mathcal{L}(d)$  let  $F_t \subset X$  be the corresponding section. We write  $Gr(1, \mathcal{L}(d))$  for the Grassmannian variety of lines in  $\mathcal{L}(d)$ .
- (2) A pencil  $\{F_t\}_{t \in L}$  of hypersurface sections parametrized by  $L \in Gr(1, \mathcal{L}(d))$ , is admissible for  $(X, C)$  if  $\Delta_L \cap C = \emptyset$  for the ax  $\Delta_L$  of  $L$  and  $F_t \cap C$  for almost all  $t \in L$ .

By [SGA7, XVIII 6.6.1], for a sufficiently large  $d$ , there always exist  $L \in \mathcal{L}(d)$  admissible for  $(X, C)$ .

Now we start the proof of Theorem 4.5. In what follows we fix  $L \in \mathcal{L}(d)$  admissible for  $(X, C)$  and  $\pi \in k(X)$  such that:

$$(7.1) \quad \operatorname{div}_X(\pi) = C + G_0 - G_\infty \quad \text{with } G_0, G_\infty \in \operatorname{Div}(X, C)^+.$$

We also fix a finite set  $T_L \subset L$  such that

$$(7.2) \quad F_t \cap C \quad \text{and} \quad F_t \cap C \cap (G_0 \cup G_\infty) = \emptyset \quad \text{for } t \in L - T_L.$$

We have the rational map

$$h_L : X \cdots \rightarrow L ; x \rightarrow t \text{ such that } x \in F_t.$$

By Definition 7.2(2)  $h_L$  is defined at any point of  $C$  and it gives rise to

$$\mathcal{O}_{L,t} \hookrightarrow \mathcal{O}_{X,x} \quad \text{for } t \in L \text{ and } x \in F_t \cap C.$$

**Lemma 7.3.** For each  $t \in L$ , choose a prime element  $f_t \in \mathcal{O}_{L,t}$ .

- (1) For  $t \in L - T_L$  and  $x \in F_t \cap C$ , we have

$$\Omega_{X,x}^1 = \mathcal{O}_{X,x} \cdot d\pi \oplus \mathcal{O}_{X,x} \cdot df_t.$$

- (2) There exists an effective divisor  $\theta$  on  $L$  independent of the choice of  $f_t$  such that  $|\theta| = T_L$  and that for any  $t \in T_L$  and  $x \in F_t \cap C$  and for any

$$\omega = \frac{1}{f_t}(\xi_1 d\pi + \xi_2 df_t) \in \Omega_X^1 \otimes_{\mathcal{O}_X} k(C) \quad \text{with } \xi_i \in k(C) = \prod_{\lambda \in I} k(C_\lambda),$$

we have the implication

$$\omega \in \Omega_X^1(-F_\theta) \otimes_{\mathcal{O}_X} \mathcal{O}_{C,x} \Rightarrow \xi_i \in \mathcal{O}_{C,x}(-F_t),$$

$$\text{where } F_\theta = \sum_{t \in T_L} e_t F_t \text{ for } \theta = \sum_{t \in T_L} e_t t \text{ with } e_t \in \mathbb{Z}_{\geq 1}.$$

*Proof.* (1) follows from the fact that  $(\pi, f_t)$  is a system of regular parameters of  $\mathcal{O}_{X,x}$  if  $t \in L - T_L$  and  $x \in F_t \cap C$ . To show (2), note

$$\Omega_X^1 \otimes_{\mathcal{O}_X} k(C) = k(C) \cdot \frac{d\pi}{f_t} \oplus k(C) \cdot \frac{df_t}{f_t}$$

and put

$$\Theta_x = \mathcal{O}_{C,x} \cdot \frac{d\pi}{f_t} \oplus \mathcal{O}_{C,x} \cdot \frac{df_t}{f_t} \subset \Omega_X^1 \otimes_{\mathcal{O}_X} k(C).$$

We see that  $\Theta_x$  is independent of the choice  $f_t$ , namely for  $f'_t = u f_t$  with  $u \in \mathcal{O}_{X,x}^\times$ ,

$$\Theta_x = \mathcal{O}_{C,x} \cdot \frac{d\pi}{f'_t} \oplus \mathcal{O}_{C,x} \cdot \frac{df'_t}{f'_t}.$$

Thus (2) follows from the fact that there exists an effective divisor  $\theta$  on  $L$  such that  $|\theta| = T_L$  and that for any  $t \in T_L$  and  $x \in F_t \cap C$ , we have

$$\Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{O}_{C,x}(-F_\theta) \subset \mathcal{O}_{C,x}(-F_t) \cdot \Theta_x.$$

□

We fix  $\theta$  as in Lemma 7.3 and put

$$(7.3) \quad \Xi = F_\theta + G_\infty \in \text{Div}(X, C)^+.$$

Taking  $d$  in Definition 7.2(1) large enough, we may assume

(♣1) For any  $t \in L - T_L$ , the map induced by (4.7):

$$\bigoplus_{x \in F_t \cap C} \frac{\Omega_X^1(-D + F_t) \otimes \mathcal{O}_{C,x}}{\Omega_X^1(-D) \otimes \mathcal{O}_{C,x}} \rightarrow H^1(C, \Omega_X^1(-D - \Xi) \otimes \mathcal{O}_C)$$

is surjective.

For the moment we assume  $k$  is finite. The modified argument to treat the general case will be given later. Let  $\iota : L \simeq \mathbb{P}_k^1 = \text{Proj}(k[T_0, T_1])$  be an isomorphism over  $k$ . For a finite extension  $\mathbb{F}_q$  of  $k$  with  $q = p^N$ , put

$$L(\mathbb{F}_q)_\iota^o = \{t \in L(\mathbb{F}_q) \mid \iota(t) \neq 0, \infty\}, \text{ where } 0 = (1 : 0), \infty = (0 : 1) \in \mathbb{P}_k^1.$$

Fix  $t_0 \in L(k) - T_L$ . Take  $\iota$  and  $q$  such that  $t_0 \in L(\mathbb{F}_q)_\iota^o$  and consider the map

$$(7.4) \quad \begin{array}{ccc} \bigoplus_{t \in L(\mathbb{F}_q)_\iota^o \setminus T_L} \bigoplus_{x \in F_t \cap C} \frac{\Omega_X^1(-D + F_t) \otimes \mathcal{O}_{C,x}}{\Omega_X^1(-D) \otimes \mathcal{O}_{C,x}} & \xrightarrow{\psi_{L,\iota,q}} & H^1(C, \Omega_X^1(-D - \Xi) \otimes \mathcal{O}_C) \\ \downarrow \sum \mu_x & & \\ W(U)/F^{(D+C)}W(X, C) & & \end{array}$$

where  $\psi_{L,\iota,q}$  is induced by (4.7). We see

$$\text{Ker}(\psi_{L,\iota,q}) = \text{Image}\left(H^0(C, \Omega_X^1(-D - \Xi + \sum_{t \in L(\mathbb{F}_q)_\iota^o \setminus T_L} F_t) \otimes \mathcal{O}_C)\right).$$

**Claim 7.4.** *Let  $\omega \in H^0(C, \Omega_X^1(-D - \Xi + \sum_{t \in L(\mathbb{F}_q)_\iota^o \setminus T_L} F_t) \otimes \mathcal{O}_C)$  and  $\omega_x$  be the image of  $\omega$  in  $\Omega_X^1(-D + F_t) \otimes \mathcal{O}_{C,x}$  for  $x \in C - T_L$ . For any  $n > 0$  we have*

$$\sum_{t \in L(\mathbb{F}_q)_\iota^o \setminus T_L} \sum_{x \in F_t \cap C} \mu_x(\omega_x) \in \widehat{F}^{(D+C)}W(X, C) + p^n \widehat{F}^{(1)}W(U).$$

We show Theorem 4.5 assuming the claim. The claim implies the existence of a map

$$\phi_{L,\iota,q} : H^1(C, \Omega_X^1(-D - \Xi) \otimes \mathcal{O}_C) \rightarrow W(U)/\widehat{F}^{(D+C)}W(X, C) + p^n \widehat{F}^{(1)}W(U)$$

such that the following diagram commutes

$$(7.5) \quad \begin{array}{ccc} \bigoplus_{t \in L(\mathbb{F}_q)_\iota^o \setminus T_L} \bigoplus_{x \in F_t \cap C} \frac{\Omega_X^1(-D + F_t) \otimes \mathcal{O}_{C,x}}{\Omega_X^1(-D) \otimes \mathcal{O}_{C,x}} & \xrightarrow{\psi_{L,\iota,q}} & H^1(C, \Omega_X^1(-D - \Xi) \otimes \mathcal{O}_C) \\ \downarrow \sum \mu_x & \searrow \phi_{L,\iota,q} & \\ W(U)/\widehat{F}^{(D+C)}W(X, C) + p^n \widehat{F}^{(1)}W(U) & & \end{array}$$

Since  $t_0 \in L(\mathbb{F}_q)_\iota^o \setminus T_L$ , (♣1) for  $t = t_0$  implies that  $\phi_L = \phi_{L,\iota,q}$  is independent of  $\iota$  and  $q$ . Take any  $x \in C - \Xi$ . Note  $t = h_L(x) \notin T_L$  (cf. (7.3) and Lemma 7.3(2)). Moreover we can choose such  $\iota$  and  $q$  that  $t \in L(\mathbb{F}_q)_\iota^o$ . Then (7.5) implies that  $\phi_L$  satisfies (4.16), which completes the proof of Theorem 4.5 in case  $k$  is finite.

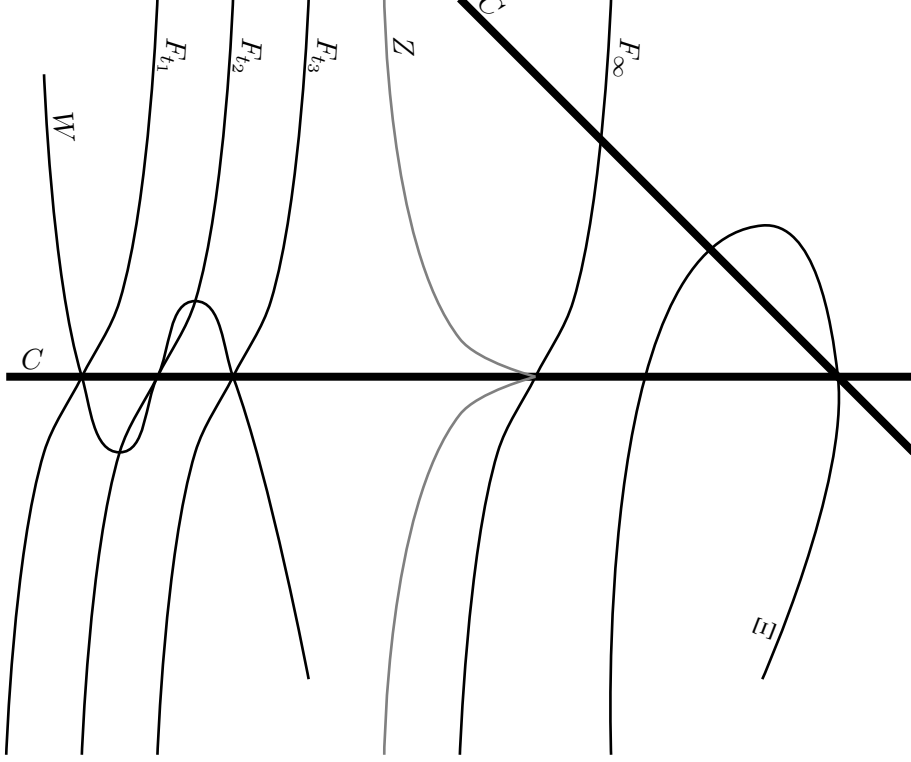


FIGURE 1. The configuration of curves in the proof of Claim 7.4 ( $t_1, t_2, t_3 \in L(\mathbb{F}_q)_\iota^\circ$ ).

Now we explain how to modify the above argument to treat the general case where  $k$  is not necessarily finite. Let  $\mathbb{F}_p$  be the prime subfield of  $k$ . For  $\iota : L \simeq \mathbb{P}_k^1 = \text{Proj}(k[T_0, T_1])$ , an isomorphism over  $k$ , let

$$g_\iota : L \rightarrow \mathbb{P}_{\mathbb{F}_p}^1$$

be the composite of  $\iota$  and  $\mathbb{P}_k^1 \rightarrow \mathbb{P}_{\mathbb{F}_p}^1$ . For  $q = p^N$  with  $N > 0$ , put

$$L(\mathbb{F}_q)_\iota = \{t \in L \mid g_\iota(t) \text{ is a } \mathbb{F}_q\text{-rational point of } \mathbb{P}_{\mathbb{F}_p}^1\} \subset L,$$

$$L(\mathbb{F}_q)_\iota^\circ = \{t \in L(\mathbb{F}_q)_\iota \mid g_\iota(t) \neq 0, \infty\}, \text{ where } 0 = (1 : 0), \infty = (0 : 1) \in \mathbb{P}_{\mathbb{F}_p}^1.$$

By Lemma 7.1, we may assume  $k \supset \mathbb{F}_q$  with  $q \geq 4$  so that

(♣2) for  $t, t' \in L(k)$ , there exists  $\iota$  such that  $t, t' \in L(\mathbb{F}_q)_\iota^\circ$ .

With this modification, we follow the same argument as above until we show that Claim 7.4 implies that  $\phi_{L, \iota, q}$  is independent of  $\iota$  and  $q$ . To show (4.16), take  $x \in C - \Xi$ . By Lemma 7.1 we may assume  $x \in F_t \cap C$  for some  $t \in L(k) - T_L$ . Then, by (♣2), we may assume  $t \in L(\mathbb{F}_q)_\iota^\circ \setminus T_L$ . Then (4.16) follows from (7.5). This completes the proof of Theorem 4.5.

*Proof of Claim 7.4.* For  $t \in L(\mathbb{F}_q)_\iota^\circ \setminus T_L$  and  $x \in F_t \cap C$ , let  $\omega_{x, C}$  be the image of  $\omega_x$  under the map

$$\Omega_X^1(-D + F_t) \otimes \mathcal{O}_{C, x} \rightarrow \Omega_{C, x}^1(-D + F_t).$$

In what follows we let  $0, \infty$  denote the closed points of  $L$  which correspond to  $0, \infty \in \mathbb{P}_{\mathbb{F}_p}^1$  by  $\iota : L \simeq \mathbb{P}_k^1$ . By Lemma 7.1 we may assume that the algebraic closure  $k_0$  of  $\mathbb{F}_p$  in  $k$  is large enough so that after a coordinate transformation of  $\mathbb{P}_{k_0}^1$ , we have

(\*1)  $0, \infty \notin T_L$ , and  $\text{ord}_x(\omega_{x, C}) = 0$  for any  $x \in C \cap F_\infty$ .

Here we assume without loss of generality that the restriction of  $\omega$  to any generic point of  $C$  is not zero. Put

$$(7.6) \quad X = T_1/T_0 \in \mathbb{F}_p(\mathbb{P}^1) \quad \text{and} \quad b = 1 - \frac{1}{X^{q-1}},$$

which are considered as elements of  $k(L) \subset k(X)$  via  $g_t$ . Put (cf. (7.1))

$$(7.7) \quad W_b = \operatorname{div}_X(b + \pi) + G_\infty + (q-1)F_0.$$

Note

$$(7.8) \quad \operatorname{div}_X(b) = \sum_{t \in L(\mathbb{F}_q)_t^o} F_t - (q-1) \cdot F_0,$$

**Claim 7.5.** (1)  $W_b \in \operatorname{Div}(X, C)^+$  and  $W_b \cap C \subset \bigcup_{t \in L(\mathbb{F}_q)_t^o} F_t \cap C$ .

(2) For  $t \in L(\mathbb{F}_q)_t^o \setminus T_L$  and  $x \in F_t \cap C$ ,  $W_b \cap C$  at  $x$  and  $\operatorname{div}_{X,x}(b + \pi)$  is the irreducible component of  $W_b$  containing  $x$ .

(3) For  $t \in L(\mathbb{F}_q)_t^o$  and  $x \in F_t \cap C$ , we have

$$\mathcal{O}_{W_b,x}(-F_t - G_\infty) \subset \mathcal{O}_{W_b,x}(-C).$$

*Proof.* (1) and (2) follow immediately from (7.1) and (7.2) and (7.8) except that  $W_b \cap F_0 \cap C = \emptyset$ , which holds since  $C = \operatorname{div}_X(\pi)$  and  $W_b = \operatorname{div}_X(X^{q-1} - 1 + \pi X^{q-1})$  locally at  $F_0 \cap C$ . The last fact is checked by noting  $b + \pi = \frac{X^{q-1} - 1 + \pi X^{q-1}}{X^{q-1}}$ . To show (3), let  $\pi_\infty$  be a local parameter of  $G_\infty$  at  $x$  if  $x \in G_\infty$  and  $\pi_\infty = 1$  otherwise. Putting  $\pi' = \pi\pi_\infty$ ,  $\pi' \mathcal{O}_{W_b,x} \subset \mathcal{O}_{W_b,x}(-C)$  by (7.1). Noting  $W_b \cap F_0 \cap C = \emptyset$ , (7.7) implies  $W_b = \operatorname{div}_X(\pi_\infty b + \pi')$  locally at  $x$ . Hence we get

$$\mathcal{O}_{W_b,x}(-F_t - G_\infty) = b\pi_\infty \mathcal{O}_{W_b,x} = \pi' \mathcal{O}_{W_b,x} \subset \mathcal{O}_{W_b,x}(-C).$$

This completes the proof of Claim 7.5.  $\square$

Write

$$\omega = \frac{1}{b}(\alpha d\pi + \beta db) \quad \text{with } \alpha, \beta \in k(C).$$

Noting (7.8) and

$$(7.9) \quad \frac{db}{b} = \frac{\rho^{q-2} d\rho}{1 - \rho^{q-1}} = \frac{dX}{X - X^q} \quad \text{with } \rho = 1/X,$$

the assumption on  $\omega$  in Claim 7.4 and Lemma 7.3 imply

$$\begin{aligned} \alpha &\in H^0(C, \mathcal{O}_C(-D + (q-1)F_0 - \sum_{t \in T_L} F_t - G_\infty)) \\ \beta &\in H^0(C, \mathcal{O}_C(-D + (q-2)F_\infty - \sum_{t \in T_L \cup \{0\}} F_t - G_\infty)). \end{aligned}$$

Since  $F_0$  and  $F_\infty$  are ample divisors on  $X$ , the maps

$$H^0(X, \mathcal{O}_X(-D + (q-1)F_0 - \sum_{t \in T_L} F_t - G_\infty)) \rightarrow H^0(C, \mathcal{O}_C(-D + (q-1)F_0 - \sum_{t \in T_L} F_t - G_\infty)),$$

$$H^0(X, \mathcal{O}_X(-D + (q-2)F_\infty - \sum_{t \in T_L \cup \{0\}} F_t - G_\infty)) \rightarrow H^0(C, \mathcal{O}_C(-D + (q-2)F_\infty - \sum_{t \in T_L \cup \{0\}} F_t - G_\infty))$$

are surjective for  $q$  sufficiently large so that we can take

$$\begin{aligned} \tilde{\alpha} &\in H^0(X, \mathcal{O}_X(-D + (q-1)F_0 - \sum_{t \in T_L} F_t - G_\infty)), \\ \tilde{\beta} &\in H^0(X, \mathcal{O}_X(-D + (q-2)F_\infty - \sum_{t \in T_L \cup \{0\}} F_t - G_\infty)), \end{aligned}$$

such that  $\omega = \tilde{\omega} \otimes k(C)$  with

$$\tilde{\omega} = \frac{1}{b}(\tilde{\alpha}d\pi + \tilde{\beta}db) \in \Omega_X^1(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{X,C},$$

where  $\mathcal{O}_{X,C}$  is the semi-local ring of  $X$  at the generic points of  $C$ . By Claim 7.5(2), for  $t \in L(\mathbb{F}_q)_i^o \setminus T_L$  and  $x \in F_t \cap C$ , we have

$$\mu_x(\omega_x) = \{1 + \tilde{\beta}\}_{F_t, x} - \{1 + \tilde{\alpha}\}_{F_t, x} + \{1 + \tilde{\alpha}\}_{W_b, x}.$$

Hence Claim 7.4 follows from the following.

**Claim 7.6.** *We have*

$$(7.10) \quad \sum_{t \in L(\mathbb{F}_q)_i^o \setminus T_L} \{1 + \tilde{\beta}\}_{F_t} \in \widehat{F}^{(D+C)}W(X, C) + p^n \widehat{F}^{(1)}W(U),$$

$$(7.11) \quad \sum_{t \in L(\mathbb{F}_q)_i^o \setminus T_L} (\{1 + \tilde{\alpha}\}_{F_t} - \sum_{x \in F_t \cap C} \{1 + \tilde{\alpha}\}_{W_b, x}) \in F^{(D+C)}W(X, C).$$

To show (7.10), first note that the condition  $(*)1$  implies  $\tilde{\beta} \in \mathcal{P}_{D, q-2}(F_\infty)$ . Noting

$$\operatorname{div}_X(1 + \tilde{\beta}) = Z_{\tilde{\beta}} - (q-2)F_\infty \quad (\text{cf. Definition 6.2}),$$

$$(1 + \tilde{\beta})|_{F_t} = 1 \quad \text{for } t \in T_L \cup \{0\} \quad \text{and} \quad b|_{F_\infty} = 1,$$

(7.8) and Lemma 1.14 imply

$$0 = \partial\{1 + \tilde{\beta}, b\} = \sum_{t \in L(\mathbb{F}_q)_i^o \setminus T_L} \{1 + \tilde{\beta}\}_{F_t} + \{1 - \rho^{q-1}\}_{Z_{\tilde{\beta}}} \quad (\rho = 1/X).$$

We have  $\{1 - \rho^{q-1}\}_{Z_{\tilde{\beta}}} \in \widehat{F}^{(D+C)}W(X, C) + p^n \widehat{F}^{(1)}W(U)$  by Lemma 6.3(2) and this proves (7.10).

To show (7.11), put

$$Z' = \operatorname{div}_X(1 + \tilde{\alpha}) + (q-1)F_0 \in \operatorname{Div}(X, C)^+.$$

Letting  $\pi_D$  be a local parameter of  $D$  at  $F_0 \cap C$ , we have

$$(7.12) \quad Z' \cap C \subset F_0 \cap C \quad \text{and} \quad Z' = \operatorname{div}_X(X^{q-1} + \pi_D \gamma) \quad \text{locally at } F_0 \cap C$$

where  $\gamma \in \mathcal{O}_{X, F_0 \cap C}$ . By (7.7) and (7.8),

$$\operatorname{div}_X\left(\frac{b + \pi}{b}\right) = W_b - G_\infty - \sum_{t \in L(\mathbb{F}_q)_i^o} F_t.$$

Noting

$$(1 + \tilde{\alpha})|_{G_\infty} = 1 \quad \text{and} \quad (1 + \tilde{\alpha})|_{F_t} = 1 \quad \text{for } t \in T_L,$$

Lemma 1.14 implies

$$(7.13) \quad 0 = \partial\left\{1 + \tilde{\alpha}, \frac{b + \pi}{b}\right\} = \{1 + \tilde{\alpha}\}_{W_b} - \sum_{t \in L(\mathbb{F}_q)_i^o \setminus T_L} \{1 + \tilde{\alpha}\}_{F_t} \\ + \left\{\frac{b + \pi}{b}\right\}_{Z'} - (q-1)\left\{\frac{b + \pi}{b}\right\}_{F_0}.$$

We claim

$$\{1 + \tilde{\alpha}\}_{W_b} - \sum_{t \in L(\mathbb{F}_q)_i^o \setminus T_L} \sum_{x \in F_t \cap C} \{1 + \tilde{\alpha}\}_{W_b, x} \in F^{(D+C)}W(X, C).$$

Indeed, by Claim 7.5(1),

$$\{1 + \tilde{\alpha}\}_{W_b} = \sum_{t \in L(\mathbb{F}_q)_i^o} \sum_{x \in F_t \cap C} \{1 + \tilde{\alpha}\}_{W_b, x}.$$

For  $t \in T_L$  and  $x \in F_t \cap C$ , we have  $\tilde{\alpha}|_{W_b} \in \mathcal{O}_{W_b, x}(-D - F_t - G_\infty)$  since  $\tilde{\alpha} \in H^0(X, \mathcal{O}_X(-D - \sum_{t \in T_L} F_t - G_\infty + (q-1)F_0))$  and  $W_b \cap F_0 \cap C = \emptyset$  by Claim 7.5(1).

Hence we have  $\{1 + \tilde{\alpha}\}_{W_b, x} \in F^{(D+C)}W(X, C)$  by Claim 7.5(3).

By the above claim we are reduced to showing that the last two terms of (7.13) belong to  $F^{(D+C)}W(X, C)$ . The assertion follows from the fact

$$\frac{b + \pi}{b} = 1 - \frac{\pi X^{q-1}}{1 - X^{q-1}}.$$

and that we have in view of (7.12),

$$\left(\frac{b + \pi}{b}\right)|_{Z'} = 1 + \frac{\gamma \pi \pi_D}{1 - X^{q-1}} \in 1 + \mathcal{O}_{Z', Z' \cap C}(-D - C).$$

This completes the proof of (7.11) and that of Theorem 4.5.  $\square$

*Remark 7.7.* If  $D \geq 2C$ , we may use Lemma 6.3(1) for the proof of (7.10) to remove  $p^n \widehat{F}^{(1)}W(U)$  out of the conclusion of Claim 7.4. This explains the case (i) of Remark 4.6.

## 8. PROOF OF KEY THEOREM II

In this section we prove Theorem 4.7. We need some preliminaries.

Let  $(A, \mathfrak{m}_A)$  be an excellent regular henselian two-dimensional local domain with the quotient field  $K$ . Assume  $F = A/\mathfrak{m}_A$  is finite. Let  $P$  be the set of prime ideals of height one in  $A$ . For  $\mathfrak{p} \in P$  let  $A_{\mathfrak{p}}$  be the henselization of  $A$  at  $\mathfrak{p}$  and  $K_{\mathfrak{p}}$  (resp.  $k(\mathfrak{p})$ ) be the quotient (resp. residue) field of  $A_{\mathfrak{p}}$ . Let  $D \subset A$  be a non-zero invertible ideal (which is identified with an effective Cartier divisor on  $\text{Spec}(A)$ ) and  $P_D \subset P$  be the subset of  $\mathfrak{p}$  dividing  $D$ . For  $\lambda \in P_D$  let  $m_\lambda \in \mathbb{Z}_{>0}$  be the multiplicity of  $\lambda$  in  $D$ . Let  $U = \text{Spec}(A[D^{-1}])$  and put

$$\text{fil}_D H^1(U) = \text{Ker}(H^1(U) \rightarrow \bigoplus_{\lambda \in P_D} H^1(K_\lambda) / \text{fil}_{m_\lambda} H^1(K_\lambda)).$$

We introduce an idele class group which describes  $\text{fil}_D H^1(U)$ :

$$(8.1) \quad C^{KS}(A, D) = \text{Coker}(K_2(K) \xrightarrow{\partial = (\partial_{\mathfrak{p}}, \partial_\lambda)} \bigoplus_{\mathfrak{p} \in P - P_D} k(\mathfrak{p})^\times \oplus \bigoplus_{\lambda \in P_D} K_2(K_\lambda) / V^{m_\lambda} K_2(K_\lambda)),$$

where  $\partial_{\mathfrak{p}}$  for  $\mathfrak{p} \notin P_D$  is the tame symbol and  $\partial_\lambda$  for  $\lambda \in P_D$  is the map induced by  $K \rightarrow K_\lambda$ . By the reciprocity law for  $A$  (cf. [Sa]) we have a canonical map

$$\Psi_U^{KS} : \text{fil}_D H^1(U) \rightarrow \text{Hom}(C^{KS}(A, D), \mathbb{Q}/\mathbb{Z})$$

such that the following diagrams are commutative for  $\mathfrak{p} \notin P_D$  and  $\lambda \in P_D$ :

$$\begin{array}{ccc} \text{fil}_D H^1(U) & \xrightarrow{\Psi_U^{KS}} & \text{Hom}(C^{KS}(A, D), \mathbb{Q}/\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^1(k(\mathfrak{p})) & \xrightarrow{\Psi_{k(\mathfrak{p})}} & \text{Hom}(k(\mathfrak{p})^\times, \mathbb{Q}/\mathbb{Z}), \\ \\ \text{fil}_D H^1(U) & \xrightarrow{\Psi_U^{KS}} & \text{Hom}(C^{KS}(A, D), \mathbb{Q}/\mathbb{Z}) \\ \downarrow & & \downarrow \\ \text{fil}_{m_\lambda} H^1(K_\lambda) & \xrightarrow{\Psi_{K_\lambda}} & \text{Hom}(K_2(K_\lambda) / V^{m_\lambda} K_2(K_\lambda), \mathbb{Q}/\mathbb{Z}) \end{array}$$

where  $\Psi_{k(\mathfrak{p})}$  (resp.  $\Psi_{K_\lambda}$ ) is the map (2.4) for the 1-dimensional (resp. 2-dimensional) local field  $k(\mathfrak{p})$  (resp.  $K_\lambda$ ).

Now we assume  $D = \lambda^m$  ( $m \in \mathbb{Z}_{\geq 1}$ ) for  $\lambda = (\pi) \in P$  such that  $B_\lambda = A/(\pi)$  is regular. Let  $\mathfrak{m}_\lambda = \mathfrak{m}_A B_\lambda$  be the maximal ideal of  $B_\lambda$ . Define

$$\nu_A : \pi^{m-1} \Omega_A^1 \otimes_A \mathfrak{m}_\lambda^{-1} \rightarrow C^{KS}(A, D)$$

as the composite

$$\pi^{m-1} \Omega_A^1 \otimes_A \mathfrak{m}_\lambda^{-1} \hookrightarrow \pi^{m-1} \Omega_{A_\lambda}^1 \otimes_{A_\lambda} k(\lambda) \xrightarrow{\rho_{K_\lambda}^m} K_2(K_\lambda)/V^m K_2(K_\lambda) \rightarrow C^{KS}(A, D),$$

where  $\rho_{K_\lambda}^m$  is the map from Lemma 2.5.

**Lemma 8.1.** (1) *The image of the composite*

$$\text{rar}_A : \text{fil}_m H^1(U) \rightarrow \text{fil}_m H^1(K_\lambda) \xrightarrow{\text{rar}_{K_\lambda}} \frac{1}{\pi^m} \Omega_{A_\lambda}^1 \otimes_{A_\lambda} k(\lambda)$$

*is contained in  $\frac{1}{\pi^m} \Omega_A^1 \otimes_A B_\lambda$  and the following diagram is commutative.*

$$\begin{array}{ccc} \text{fil}_m H^1(U) & \xrightarrow{\text{rar}_A} & \frac{1}{\pi^m} \Omega_A^1 \otimes_A B_\lambda \\ \downarrow \Psi_U^{KS} & & \downarrow \tau_A \\ (C^{KS}(A, D))^\vee & \xrightarrow{(\nu_A)^\vee} & (\pi^{m-1} \Omega_A^1 \otimes_A \mathfrak{m}_\lambda^{-1})^\vee, \end{array}$$

*where  $\tau_A$  is induced by the pairing*

$$\begin{aligned} \frac{1}{\pi^m} \Omega_A^1 \otimes_A B_\lambda \times \pi^{m-1} \Omega_A^1 \otimes_A \mathfrak{m}_\lambda^{-1} &\rightarrow \pi^{-1} \Omega_A^2 \otimes_A \mathfrak{m}_\lambda^{-1} \xrightarrow{\text{Res}_\lambda} \mathfrak{m}_\lambda^{-1} \Omega_{B_\lambda}^1 \\ &\xrightarrow{\text{Res}_{\mathfrak{m}_\lambda}} F = B_\lambda/\mathfrak{m}_\lambda \xrightarrow{\text{Tr}_{F/\mathbb{F}_p}} \mathbb{F}_p \simeq \mathbb{Z}/p\mathbb{Z}. \end{aligned}$$

(2) *Let  $(\pi, f)$  be a system of regular parameters of  $A$  and put  $\mathfrak{p}_1 = (f)$ ,  $\mathfrak{p}_2 = (\pi + f) \in P$ . Then*

$$\xi = \nu_A\left(\frac{1}{f}(\alpha d\pi + \beta df)\right) \quad \text{for } \alpha, \beta \in (\pi^{m-1})$$

*is the image in  $C^{KS}(A, D)$  of*

$$\eta = \{1 + \beta\}_{k(\mathfrak{p}_1)} - \{1 + \alpha\}_{k(\mathfrak{p}_1)} + \{1 + \alpha\}_{k(\mathfrak{p}_2)} \in \bigoplus_{\mathfrak{p} \in P - P_D} k(\mathfrak{p})^\times.$$

*Proof.* The first (resp. second) assertion of (1) follows from [Ma, Prop.4.2.3] (resp. (2.4)(ii)). By definition,  $\xi$  is the image in  $C^{KS}(A, D)$  of

$$\{1 + \beta, f\} + \{1 + \alpha\pi/f, \pi\}, \in K_2(K_\lambda).$$

For  $\partial$  in (8.1), we compute  $\partial_{\text{tame}}(\xi) = \eta$ , where

$$\partial_{\text{tame}} = \sum_{\mathfrak{p} \notin P_D} \partial_{\mathfrak{p}}, \quad \xi = \{1 + \beta, f\} + \{1 + \alpha, \frac{f + \pi}{\pi}\} \in K_2(K).$$

Thus it suffices to show

$$\{1 + \alpha, \frac{f + \pi}{\pi}\} \equiv \{1 + \alpha\pi/f, \pi\} \in K_2(K_\lambda)/V^m K_2(K_\lambda).$$

Noting

$$\{1 + \alpha\pi/f, \pi\} = -\{1 + \alpha\pi/f, -\alpha/f\} \equiv -\{1 + \alpha\pi/f, \alpha\} \in K_2(K_\lambda)/V^m K_2(K_\lambda),$$

the assertion follows from the following equality in  $K_2(K_\lambda)$ :

$$\{1 + \alpha, \frac{f + \pi}{\pi}\} = -\{1 + (1 + \alpha)^{-1} \alpha\pi/f, -\alpha(1 + \frac{\pi}{f})\}.$$

□

Now let the notation be as in §4. In particular assume  $|D| = C$ . For a closed point  $x \in C$ , let  $C_x^{KS}(X, D) = C^{KS}(A_x, D_x)$  be defined as in (8.1), where  $A_x$  is the henselization of  $\mathcal{O}_{X,x}$  and  $D_x \subset A_x$  is the ideal for  $D \times_X \text{Spec}(A_x)$ . Let  $K_x$  be the fraction field of  $A_x$  and  $P_x$  be the set of prime ideals of height one in  $A_x$ . We introduce the idele class group for  $(X, D)$ :

$$C^{KS}(X, D) = \text{Coker}\left(\bigoplus_{Z \subset X} k(Z)^\times \xrightarrow{\partial} Z_0(U) \oplus \bigoplus_{x \in C} C_x^{KS}(X, D)\right),$$

where  $Z$  ranges over integral curves on  $X$  not contained in  $C$ , and  $\partial$  is induced by the divisor maps  $k(Z)^\times \rightarrow Z_0(U)$ . By the class field theory developed in [KS], we have a canonical map

$$\Psi_U^{KS} : H^1(U) \rightarrow \text{Hom}(C^{KS}(X, D), \mathbb{Q}/\mathbb{Z})$$

which fits into the commutative diagram for any  $x \in C$ ,

$$(8.2) \quad \begin{array}{ccc} \text{fil}_D H^1(U) & \xrightarrow{\Psi_U^{KS}} & C^{KS}(X, D)^\vee \\ \downarrow & & \downarrow \\ \text{fil}_{D_x} H^1(U_x) & \xrightarrow{\Psi_{U_x}^{KS}} & C_x^{KS}(X, D)^\vee \end{array}$$

where  $U_x = \text{Spec}(A_x) \times_X U$ . We have a canonical map

$$\epsilon_X : W(U)/F^{(D)}W(X, C) \rightarrow C^{KS}(X, D)$$

such that the following diagram commutes.

$$\begin{array}{ccc} \text{fil}_D H^1(U) & \xrightarrow{\Psi_U^{KS}} & C^{KS}(X, D)^\vee \\ \downarrow \Psi_U & & \downarrow (\epsilon_X)^\vee \\ C(X, D)^\vee & \longrightarrow & (W(U)/F^{(D)}W(X, C))^\vee \end{array}$$

To construct  $\epsilon_X$ , we first note that there is a natural map

$$\tilde{\epsilon}_X : W(U) = \text{Coker}\left(\bigoplus_{Z \subset X} k(Z)^\times \rightarrow \bigoplus_{Z \subset X} k(Z)_\infty^\times \oplus Z_0(U)\right) \rightarrow C^{KS}(X, D)$$

in view of the fact that for  $Z \subset X$  as above,

$$k(Z)_\infty^\times = \bigoplus_{x \in Z \cap C} \bigoplus_{\mathfrak{p} \in P_{x,Z}} k(\mathfrak{p}),$$

where  $P_{x,Z}$  denotes the set of  $\mathfrak{p} \in P_x$  lying over  $Z$ . The map  $\tilde{\epsilon}_X$  annihilates  $F^{(D)}W(X, C)$  by the following fact: Let  $x$  be a closed point of  $C$  and take  $\alpha \in \mathcal{O}_{X,x}(-D)$ . Let  $f \in A_x$  be such that  $Z \times_X \text{Spec}(A_x) = \text{Spec}(A_x/(f))$ . Let  $P_{x,D}$  be the set of  $\mathfrak{q} \in P_x$  dividing  $D_x$ . Then

$$\{1 + \alpha, f\} \in K_2(K_x)$$

vanishes in  $K_2(K_{\mathfrak{q}})/V^{m_{\mathfrak{q}}}K_2(K_{\mathfrak{q}})$  for any  $\mathfrak{q} \in P_{x,D}$ , where  $K_{\mathfrak{q}}$  is the henselization of  $K_x$  at  $\mathfrak{q}$  and  $m_{\mathfrak{q}}$  is the multiplicity of  $\mathfrak{q}$  in  $D_x$ . Moreover we have

$$\partial_{\text{tame}}(\{1 + \alpha, f\}) = (1 + \alpha|_Z)_{k(\mathfrak{q}) \in P_{x,Z}},$$

where  $\partial_{\text{tame}}$  is the tame part of  $\partial$  in (8.1):

$$K_2(K_x) \rightarrow \bigoplus_{\mathfrak{p} \in P_x - P_{x,D}} k(\mathfrak{p})^\times.$$

Finally Theorem 4.7 follows from the following commutative diagram for regular closed points  $x$  of  $C$ :

$$\begin{array}{ccccc}
\mathrm{fil}_D H^1(U) & \xrightarrow{\quad} & \mathrm{fil}_{D_x} H^1(U_x) & \xrightarrow{\quad} & \mathrm{fil}_{D_x} H^1(K_{\lambda_x}) \\
\downarrow \Psi_U^{KS} & \boxed{I} & \downarrow \Psi_{U_x}^{KS} & \boxed{III} & \downarrow \mathrm{rar}_{K_\lambda} \\
C^{KS}(X, D)^\vee & \xrightarrow{\quad} & C_x^{KS}(X, D)^\vee & & \Omega_X^1(D) \otimes_{\mathcal{O}_X} k(\lambda_x) \\
\downarrow (\epsilon_X)^\vee & \boxed{II} & \downarrow (\nu_{A_x})^\vee & & \uparrow \\
(W(U)/F^{(D)}W(X, C))^\vee & \xrightarrow{(\mu_x)^\vee} & (\Omega_X^1(-D + C) \otimes_{\mathcal{O}_X} \mathcal{O}_{C, x}(x))^\vee & \xleftarrow{\tau_{A_x}} & \Omega_X^1(D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C, x} \\
\uparrow & \boxed{IV} & & & \uparrow \\
C(X, D)^\vee & \xrightarrow{\quad \mathrm{cc}_X \quad} & H^0(C, \Omega_X^1(D + \Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C) & & 
\end{array}$$

Here the commutativity of  $\boxed{I}$  comes from (8.2), that of  $\boxed{II}$  from Lemma 8.1(2), that of  $\boxed{III}$  from Lemma 8.1(1), and that of  $\boxed{IV}$  from the definition of  $\mathrm{cc}_X$ . This completes the proof of Theorem 4.7.

## 9. PROOF OF KEY LEMMA I

Let the notation be as in §6. In this section we prove Lemma 6.1. First we prove the following.

**Lemma 9.1.** *Take a regular closed point  $x \in C$  and  $Z_1, Z_2 \in \mathrm{Div}(X, C)^+$  such that  $x \in Z_1 \cap Z_2$  and that  $Z_i \cap C$  at  $x$  for  $i = 1, 2$  (cf. Definition 1.1). Assume  $(Z_1, Z_2)_x \geq e + 1$  for an integer  $e \geq 1$ . Then*

$$\{1 + \alpha\}_{Z_1, x} - \{1 + \alpha\}_{Z_2, x} \in F^{(D+eC)}W(X, C) \quad \text{for } \alpha \in \mathcal{O}_{X, x}(-D).$$

*Proof.* For an integer  $d > 0$ , let  $\mathcal{L}(d) = |dH|$  be as Definition 7.2. Take  $d$  sufficiently large so that we can choose  $F$  in  $\mathcal{L}(d)$  satisfying the conditions:

- (#1)  $F \cap C$ , and  $x \in F$  and  $F \cap C \cap (Z_1 \cup Z_2 - x) = \emptyset$ , and  $F \cap Z_i$  at  $x$  for  $i = 1, 2$ .
- (#2)  $H^1(X, \mathcal{O}_X(-D - (e + 1 + i)C + (e + 1)F)) = 0$  for  $1 \leq i \leq e$ ,

$$H^1(C, \mathcal{O}_C(-D - (e + 1 + i)C + F)) = 0 \text{ for } 0 \leq i \leq e - 1,$$

Then take  $d'$  sufficiently large so that we can choose integral  $H$  in  $\mathcal{L}(d')$  satisfying

- (♠1)  $(F \cap C) - x \subset H$  and  $x \notin H$ .
- (♠2)  $H \cap C$  at  $(F \cap C) - x$  and  $H \cap F$  at  $(F \cap C) - x$ .

Then take  $d''$  sufficiently large so that we can choose  $F_i$  in  $\mathcal{L}(d'')$  for  $i = 1, 2$  satisfying

- (♠3)  $F_i = Z_i + H + G_i$  for  $G_i \in \mathrm{Div}(X, C)^+$  such that  $G_i \cap F \cap C = \emptyset$ .

Choose  $\pi, \pi_D, f \in \mathcal{O}_{X, F \cap C}$  such that locally at  $F \cap C$ ,

$$C = \mathrm{div}_X(\pi), \quad D = \mathrm{div}_X(\pi_D), \quad F = \mathrm{div}_X(f)$$

so that  $(\pi, f)$  is a system of regular parameters in  $\mathcal{O}_{X, F \cap C}$ .

**Claim 9.2.** *There are  $u_1, u_2 \in \mathcal{O}_{X, F \cap C}^\times$  such that locally at  $F \cap C$ ,*

$$F_i = \mathrm{div}_X(\pi - u_i f) \quad \text{and} \quad u_1 - u_2 \in (\pi - u_2 f, f^e) \subset \mathfrak{m}_{F \cap C},$$

where  $\mathfrak{m}_{F \cap C} = (f, \pi)$  is the radical of  $\mathcal{O}_{X, F \cap C}$ .

*Proof.* Let  $g_i \in \mathcal{O}_{X, F \cap C}$  be a local equation of  $F_i$  at  $F \cap C$ . By the assumption  $g_i \in \mathfrak{m}_{F \cap C}$  so that we can write  $g_i = a_i \pi + b_i f$  for some  $a_i, b_i \in \mathcal{O}_{X, F \cap C}$ . By the assumption,  $F_i \cap F$  and  $F_i \cap C$  at any  $y \in F \cap C$ . It implies  $a_i, b_i \in \mathcal{O}_{X, y}^\times$  which proves the first assertion. By the assumptions we have

$$(9.1) \quad \mathcal{O}_{X, F \cap C} / (\pi - u_2 f) = \mathcal{O}_{F_2, F \cap C} = \mathcal{O}_{Z_2, x} \times \mathcal{O}_{H, F \cap C - x},$$

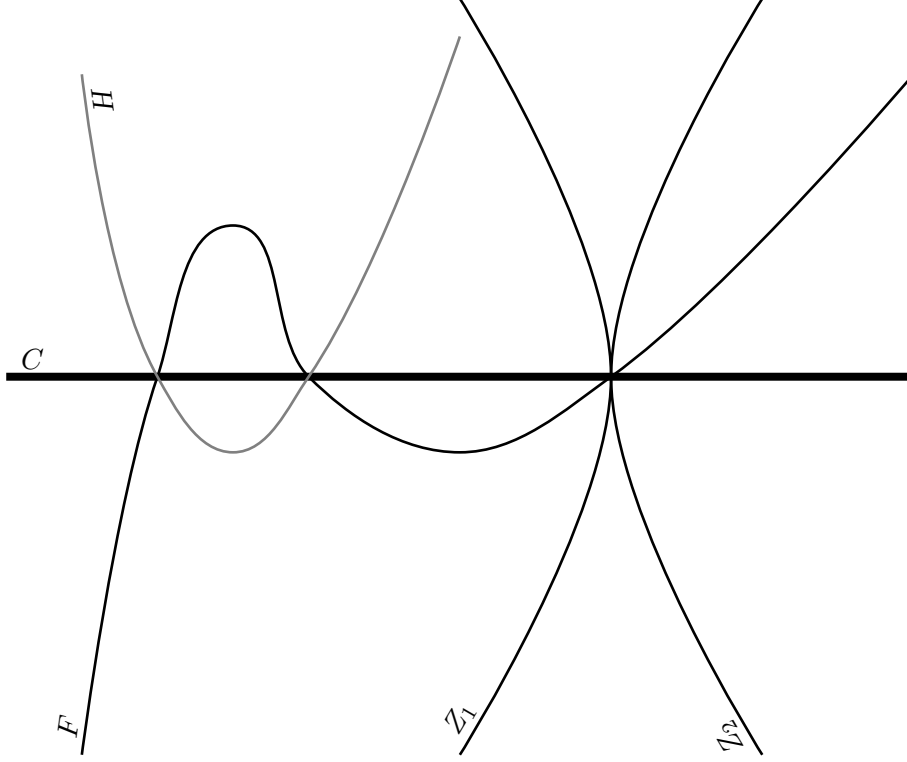


FIGURE 2. The configuration of curves in the proof of Lemma 9.1.

and

$$H = \operatorname{div}_X(\pi - u_1 f) = \operatorname{div}_X(\pi - u_2 f) \text{ locally at } F \cap C - x.$$

Hence we get  $(u_1 - u_2)f|_H = 0 \in \mathcal{O}_{H, F \cap C - x}$  and

$$(u_1 - u_2)|_H = 0 \in \mathcal{O}_{H, F \cap C - x}.$$

On the other hand the assumption implies

$$(Z_1, Z_2)_x = \operatorname{length}_{\mathcal{O}_{Z_2, x}}(\mathcal{O}_{Z_2, x}/((u_1 - u_2)f)) \geq e + 1.$$

which implies  $(u_1 - u_2)|_{Z_2} \in (f^e)\mathcal{O}_{Z_2, x}$  noting that  $f$  generates the maximal ideal of  $\mathcal{O}_{Z_2, x}$  since  $Z_2 \cap F$  at  $x$ . In view of (9.1) we get

$$(u_1 - u_2)|_{F_2} \in (f^e)\mathcal{O}_{F_2, F \cap C}.$$

which proves the second assertion of the claim.  $\square$

**Claim 9.3.** *Putting*

$$\mathcal{F} = \mathcal{O}_X(-D - (e + 1)C + (e + 1)F) \quad \text{and} \quad \mathcal{F}_{F \cap C} = \bigoplus_{y \in F \cap C} \mathcal{F}_y,$$

the natural map

$$\iota : H^0(X, \mathcal{F}) \rightarrow \mathcal{F}_{F \cap C} \otimes \mathcal{O}_{X, F \cap C}/(\pi^e, f^e)$$

is surjective.

*Proof.* The map  $H^0(X, \mathcal{F}) \rightarrow H^0(X, \mathcal{F} \otimes \mathcal{O}_X/I_C^e)$  is surjective since

$$H^1(X, \mathcal{F} \otimes I_C^e) = H^1(X, \mathcal{O}_X(-D - (2e + 1)C + (e + 1)F)) = 0$$

by the assumption (#2). For  $0 \leq i \leq e-1$  consider a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(C, \mathcal{F} \otimes I_C^i / I_C^{i+1}) & \longrightarrow & H^0(X, \mathcal{F} \otimes \mathcal{O}_X / I_C^{i+1}) & \xrightarrow{\alpha} & H^0(X, \mathcal{F} \otimes \mathcal{O}_X / I_C^i) \longrightarrow 0 \\ & & \beta \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{F}_{F \cap C} \otimes \pi^i \left( \frac{\mathcal{O}_{X, F \cap C}}{(\pi, f^e)} \right) & \longrightarrow & \mathcal{F}_{F \cap C} \otimes \frac{\mathcal{O}_{X, F \cap C}}{(\pi^{i+1}, f^e)} & \longrightarrow & \mathcal{F}_{F \cap C} \otimes \frac{\mathcal{O}_{X, F \cap C}}{(\pi^i, f^e)} \longrightarrow 0 \end{array}$$

where the surjectivity of  $\alpha$  follows from

$$H^1(C, \mathcal{F} \otimes I_C^i / I_C^{i+1}) = H^1(C, \mathcal{O}_C(-D - (e+1+i)C + (e+1)F)) = 0$$

by the assumption (#2). The map  $\beta$  is identified with

$$H^0(C, \mathcal{O}_C(-D - (e+1+i)C + (e+1)F)) \rightarrow \frac{\mathcal{O}_{C, F \cap C}(-D - (e+1+i)C + (e+1)F)}{\mathcal{O}_{C, F \cap C}(-D - (e+1+i)C + F)}$$

and it is surjective since  $H^1(C, \mathcal{O}_C(-D - (e+1+i)C + F)) = 0$  by (#2). The claim now follows by induction on  $i$ .  $\square$

There is a natural isomorphism

$$\begin{aligned} \mu : \mathcal{F}_x \otimes \frac{\mathcal{O}_{X,x}}{(\pi^e, f^e)} &\simeq \mathcal{O}_{X,x} / (\pi^e, f^e) \\ \beta \pi_D \left( \frac{\pi}{f} \right)^{e+1} &\rightarrow \beta \bmod (\pi^e, f^e) \quad (\beta \in \mathcal{O}_{X,x}). \end{aligned}$$

Take  $\alpha \in \mathcal{O}_{X,x}$ . By Claim 9.3 we can find

$$(9.2) \quad a \in H^0(X, \mathcal{O}_X(-D - (e+1)C + (e+1)F))$$

such that

$$\mu(a) = \alpha \bmod (\pi^e, f^e) \in \mathcal{O}_{X,x} / (\pi^e, f^e).$$

It implies

$$(9.3) \quad a = \gamma \pi_D \left( \frac{\pi}{f} \right)^{e+1} \quad \text{with } \gamma \in \mathcal{O}_{X,x} \text{ such that } \gamma - \alpha \in (\pi^e, f^e) \mathcal{O}_{X,x}$$

Put

$$(9.4) \quad Z' = \operatorname{div}_X(1+a) + \epsilon F \quad \text{with } \epsilon = -\operatorname{ord}_F(a) \leq e+1.$$

Then  $Z' \in \operatorname{Div}(X, C)^+$  and  $Z' \cap C \subset F \cap C$  and that locally at  $F \cap C$ ,

$$(9.5) \quad Z' = \operatorname{div}_X(f^\epsilon + \pi_D \pi^{e+1} \cdot c) \quad (c \in \mathcal{O}_{X, F \cap C}).$$

Take a rational function  $b \in k(X)^\times$  such that

$$\operatorname{div}_X(b) = F_1 - F_2.$$

By Lemma 1.14 we get

$$(9.6) \quad W(X, C) \ni 0 = \partial\{1+a, b\} = \{1+a\}_{Z_1} - \{1+a\}_{Z_2} + \{1+a\}_{G_1} - \{1+a\}_{G_2} - \epsilon\{b\}_F + \{b\}_{Z'}.$$

By (#1) we have

$$a|_{Z_i} \in \mathcal{O}_{Z_i, y}(-D - (e+1)C) \quad \text{for } y \in (Z_i \cap C) - x.$$

By (2) we have

$$a|_{G_i} \in \mathcal{O}_{G_i, G_i \cap C}(-D - (e+1)C) \quad \text{for } i = 1, 2.$$

Noting  $\pi = u_i f$  in  $\mathcal{O}_{Z_i, x}$  (cf. Claim 9.2) and that  $Z_i \cap F$  and  $Z_i \cap C$  at  $x$ , (9.3) implies

$$a|_{Z_i} = u_i^{e+1} \gamma \pi_D \equiv u_i^{e+1} \alpha \pi_D \in \frac{\mathcal{O}_{Z_i, x}(-D)}{\mathcal{O}_{Z_i, x}(-D - eC)} \quad (i = 1, 2).$$

Hence (9.6) implies

$$\{1 + u_1^{e+1} \alpha \pi_D\}_{Z_1, x} - \{1 + u_2^{e+1} \alpha \pi_D\}_{Z_2, x} - \epsilon\{b\}_F + \{b\}_{Z'} \in F^{(D+eC)}W(X, C).$$

Noting  $u_1, u_2 \in \mathcal{O}_{X, F \cap C}^\times$  and  $u_1 - u_2 \in (\pi, f) \subset \mathcal{O}_{X, x}$ , Lemma 9.1 follows from

$$(9.7) \quad -\epsilon\{b\}_F + \{b\}_{Z'} \in F^{(D+eC)}W(X, C).$$

**Claim 9.4.** *There exists  $b' \in \mathcal{O}_{X, F \cap C}^\times$  such that*

$$(9.8) \quad b'|_{Z'} = b|_{Z'}, \quad \text{and} \quad (b/b')|_F \in 1 + \mathcal{O}_{F, C \cap F}(-D - eC).$$

*Proof.* By Claim 9.2, we can write

$$b = v \frac{\pi - u_1 f}{\pi - u_2 f}, \quad u_2 - u_1 = \alpha(\pi - u_2 f) + \beta f^e,$$

where  $v \in \mathcal{O}_{X, F \cap C}^\times$  and  $\alpha, \beta \in \mathcal{O}_{X, F \cap C}$ . Then

$$v^{-1}b = 1 + \frac{(u_2 - u_1)f}{\pi - u_2 f} = 1 + \alpha f + \frac{\beta f^{e+1}}{\pi - u_2 f}.$$

Put

$$\delta = \frac{\pi^\epsilon - u_2^\epsilon f^\epsilon}{\pi - u_2 f} \in (\pi, f) \subset \mathcal{O}_{X, F \cap C}.$$

Noting (9.5) and  $\epsilon \leq e + 1$ , we compute

$$\left(\frac{\beta f^{e+1}}{\pi - u_2 f}\right)|_{Z'} = \frac{-\beta c \pi_D \pi^{e+1} f^{e+1-\epsilon} \delta}{\pi^\epsilon - u_2^\epsilon f^\epsilon} = \gamma|_{Z'},$$

where

$$\gamma = \frac{-\beta c \pi_D \pi^{e+1-\epsilon} f^{e+1-\epsilon} \delta}{1 + c u_2^\epsilon \pi_D \pi^{e+1-\epsilon}} \in \mathcal{O}_{X, F \cap C}.$$

We easily see  $\gamma \in (\pi_D \pi^e, f) \subset \mathcal{O}_{X, F \cap C}$ . On the other hand, we have  $b|_F = v$ . Hence it suffices to take  $b' = v(1 + \alpha f + \gamma)$  for Claim 9.4.  $\square$

We now deduce (9.7) from Claim 9.4. Since  $b' \in \mathcal{O}_{X, F \cap C}^\times$ , we may write  $\text{div}_X(b') = W_1 - W_2$  with  $W_i \in \text{Div}(X, C)^+$  such that  $W_i \cap F \cap C = \emptyset$  for  $i = 1, 2$ . By Lemma 1.14 and (9.4) we get

$$W(X, C) \ni 0 = \partial\{1 + a, b'\} = \{1 + a\}_{W_1} - \{1 + a\}_{W_2} - \epsilon\{b'\}_F + \{b'\}_{Z'}.$$

Since  $W_i \cap F \cap C = \emptyset$ , (9.2) implies  $a \in \mathcal{O}_{W_i, C \cap W_i}(-D - (e+1)C)$  so that

$$-\{b'\}_F + \{b'\}_{Z'} \in F^{(D+(e+1)C)}W(X, C).$$

Claim 9.4 implies

$$-\{b\}_F + \{b\}_{Z'} - (-\{b'\}_F + \{b'\}_{Z'}) = \left\{\frac{b'}{b}\right\}_F \in \{1 + \mathcal{O}_{F, C \cap F}(-D - eC)\}_F,$$

which proves (9.7).  $\square$

*Proof of Lemma 6.1:* In case  $(Z_1, Z_2)_x \geq 2$ , Lemma 6.1 follows from Lemma 9.1. We assume  $Z_1 \pitchfork Z_2$  at  $x$ . By Lemma 9.1, we may replace  $F$  by any curve which is regular at  $x$  and tangent to  $F$  at  $x$ . By Bertini's theorem we can choose  $F$  in  $\mathcal{L}(d)$  for sufficiently large  $d$  so that the following conditions hold:

- (#1)  $F \pitchfork C$  and  $F \cap C \cap (Z_1 \cup Z_2 - x) = \emptyset$ .
- (#2)  $H^1(X, \mathcal{O}_X(-D - 2C + F)) = H^1(C, \mathcal{O}_C(-D - C - x) \otimes \mathcal{O}_X(F)) = 0$ .

We may also assume

- (#3)  $C = \text{div}_X(\pi)$  and  $F = \text{div}_X(f)$  locally at  $F \cap C$ , and  $(\pi, f)$  is a system of regular parameters at any  $y \in F \cap C$ .

We then take integral  $H \in \text{Div}(X, C)^+$  satisfying

- (♠1)  $x \notin H$  and  $(F \cap C) - x \subset H$ , and  $H \pitchfork C$  and  $H \pitchfork F$  at any  $y \in (F \cap C) - x$ .

Then take  $d'$  sufficiently large so that we can choose  $F_i$  in  $\mathcal{L}(d')$  for  $i = 1, 2$  satisfying

(♠2)  $F_i = Z_i + H + G_i$  for  $G_i \in \text{Div}(X, C)^+$  such that  $G_i \cap F \cap C = \emptyset$ .

**Claim 9.5.** *The condition (#2) implies the surjectivity of the map*

$$\mu : H^0(X, \mathcal{O}_X(-D - C + F)) \rightarrow \frac{\mathcal{O}_{C,x}(-D - C + F)}{\mathcal{O}_{C,x}(-D - C + F - x)}.$$

*Proof.* Indeed the map  $\mu$  factors as

$$H^0(X, \mathcal{O}_X(-D - C + F)) \rightarrow H^0(C, \mathcal{O}_C(-D - C + F)) \rightarrow \frac{\mathcal{O}_{C,x}(-D - C + F)}{\mathcal{O}_{C,x}(-D - C + F - x)}.$$

The first (resp. second) map is surjective due to the assumption  $H^1(X, \mathcal{O}_X(-D - 2C + F)) = 0$  (resp.  $H^1(C, \mathcal{O}_C(-D - C - x) \otimes \mathcal{O}_X(F)) = 0$ ).  $\square$

Fix  $\pi_D \in \mathcal{O}_{X, F \cap C}$  such that  $D = \text{div}_X(\pi_D)$  locally at  $F \cap C$ .

**Claim 9.6.** *Locally at  $F \cap C$ , we have for  $i = 1, 2$*

$$F_i = \text{div}_X(\pi + v_i f) \quad \text{with } v_i \in \mathcal{O}_{X, F \cap C}^\times$$

and we have

$$v_i - u_i \in \mathfrak{m}_x \quad \text{for } i = 1, 2, \quad \text{and} \quad v_1 - v_2 \in \mathfrak{m}_y \quad \text{for } y \in (F \cap C) - x,$$

where  $\mathfrak{m}_y$  is the maximal ideal of  $\mathcal{O}_{X,y}$ .

*Proof.* Noting that  $(\pi, f)$  is the radical of  $\mathcal{O}_{X, F \cap C}$  by (#3), the first assertion follows from the fact that  $F \cap C \subset F_i$  and that  $F_i \cap C$  and  $F_i \cap F$  at any point of  $F \cap C$ . (#1) and (♠2) imply

$$\begin{aligned} \frac{\pi + v_i f}{\pi + u_i f} &= 1 + \frac{(v_i - u_i)f}{\pi + u_i f} \in \mathcal{O}_{X,x}^\times \quad \text{for } i = 1, 2, \\ \frac{\pi + v_1 f}{\pi + v_2 f} &= 1 + \frac{(v_1 - v_2)f}{\pi + v_2 f} \in \mathcal{O}_{X,y}^\times \quad \text{for } y \in (F \cap C) - x. \end{aligned}$$

The second assertion follows easily from this.  $\square$

By Claim 9.5, for a given  $\alpha \in \mathcal{O}_{X,x}(-D)$ , we can take

$$(9.9) \quad a \in H^0(X, \mathcal{O}_X(-D - C + F)) \quad \text{such that } a - \alpha \frac{\pi}{f} \in \mathfrak{m}_x \mathcal{O}_{X,x}(-D - C + F),$$

where  $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$  is the maximal ideal. Put

$$Z' = \text{div}_X(1 + a) + F \in \text{Div}(X, C)^+.$$

We have  $Z' \cap C \subset F \cap C$  and (9.9) implies that locally at  $F \cap C$ ,

$$(9.10) \quad Z' = \text{div}_X(f + \beta \pi) \quad \text{where } \beta = af/\pi \in \mathcal{O}_{X, F \cap C}(-D) \text{ and } \beta - \alpha \in \mathfrak{m}_x \mathcal{O}_{X,x}(-D).$$

By the construction there is  $b \in k(X)^\times$  such that

$$\text{div}_X(b) = F_1 - F_2.$$

By Lemma 1.14 we have

$$0 = \partial\{1 + a, b\} = \{1 + a\}_{Z_1} - \{1 + a\}_{Z_2} + \{1 + a\}_{G_1} - \{1 + a\}_{G_2} - \{b\}_F + \{b\}_{Z'}.$$

We have

$$\begin{aligned} a|_{G_i} &\in \mathcal{O}_{G_i, G_i \cap C}(-D - C) \quad \text{since } G_i \cap F \cap C = \emptyset, \\ a|_{Z_i} &\in \mathcal{O}_{Z_i, y}(-D - C) \text{ for } y \in Z_i \cap C - x \quad \text{since } (Z_i - x) \cap F \cap C = \emptyset, \\ a|_{Z_i} &\equiv u_i \alpha|_{Z_i} \in \frac{\mathcal{O}_{Z_i, x}(-D)}{\mathcal{O}_{Z_i, x}(-D - x)} = \frac{\mathcal{O}_{Z_i, x}(-D)}{\mathcal{O}_{Z_i, x}(-D - C)} \quad \text{by Claim 9.6 and (9.9).} \end{aligned}$$

Hence we get

$$\{1 + \alpha\}_{Z_{1,x}} - \{1 + \alpha\}_{Z_{2,x}} - \{b\}_F + \{b\}_{Z'} \in F^{(D)}W(X, C)$$

and we are reduced to showing

$$(9.11) \quad \{1 - (u_1 - u_2)\alpha\}_{F,x} - \{b\}_F + \{b\}_{Z'} \in F^{(D+C)}W(X, C).$$

By the same argument which deduced (9.7) from Claim 9.4, (9.11) follows from the following.

**Claim 9.7.** *There exists  $b' \in \mathcal{O}_{X, F \cap C}^\times$  such that  $b'|_{Z'} = b|_{Z'}$  and that*

$$(b'/b)|_F - 1 \in \mathcal{O}_{F, F \cap C}(-D) \cap \mathcal{O}_{F, (F \cap C) - x}(-D - C),$$

$$(b'/b)|_F - 1 \equiv -(u_1 - u_2)\alpha \in \frac{\mathcal{O}_{F,x}(-D)}{\mathcal{O}_{F,x}(-D - x)} = \frac{\mathcal{O}_{F,x}(-D)}{\mathcal{O}_{F,x}(-D - C)}.$$

*Proof.* By Claim 9.6 we can write

$$b = v \frac{\pi + v_1 f}{\pi + v_2 f} \quad \text{with } v \in \mathcal{O}_{X, F \cap C}^\times$$

and  $b|_F = v$ . Noting  $f + \beta\pi = 0$  on  $Z'$ , we compute

$$v^{-1}b|_{Z'} = 1 - \frac{(v_1 - v_2)\beta}{1 - \beta}$$

Noting  $\beta \in \mathcal{O}_{X, F \cap C}(-D)$ ,  $b' = v(1 - \frac{(v_1 - v_2)\beta}{1 - \beta})$  satisfies the desired properties of the claim by (9.9) and (9.10).  $\square$

## 10. PROOF OF KEY LEMMA II

Lemma 6.3 is an immediate consequence of the following.

**Lemma 10.1.** *Let the assumption be as in Lemma 6.3.*

(1) *Assuming  $e \geq m_\lambda$ , we have*

$$\{1 + f^{e+1}\mathcal{O}_{Z_a, F \cap C}\}_{Z_a, x} \subset F_B^{(2D-C)}W(X, C).$$

(2) *Assuming  $(p, 2m_\lambda) = 1$  and  $e \geq m_\lambda(p^n - 1)$  for a given integer  $n > 0$ , we have*

$$\{1 + f^{e+1}\mathcal{O}_{Z_a, F \cap C}\}_{Z_a, x} \subset \widehat{F}^{(2D)}W(X, C) + p^n \widehat{F}^{(1)}W(U).$$

We start proving Lemma 10.1. In this section we complete the proof of Lemma 10.1(1). The proof of (2) will be complete in §12.

**Lemma 10.2.** *Let the notation be as Definition 6.2. Fix integers  $\epsilon \leq e - 1$  and  $i \geq 1$ . Assume*

$$(10.1) \quad H^1(X, \mathcal{O}_X(-2D - C + \epsilon F)) = H^1(C, \mathcal{O}_C(-2D + F)) = 0.$$

*For  $a \in \mathcal{P}_{D, \epsilon}(F)$ , we have*

$$\{1 + \frac{\pi_D^2}{f^\epsilon}\mathcal{O}_{Z_a, F \cap C}\}_{Z_a} \subset \{1 + \frac{\pi_D^2 \pi}{f^\epsilon}\mathcal{O}_{Z_a, F \cap C}\}_{Z_a} + \{1 + \frac{\pi_D^2}{f}\mathcal{O}_{Z_a, F \cap C}\}_{Z_a} + F^{(2D)}W(X, C).$$

*Proof.* We fix  $a \in \mathcal{P}_{D, \epsilon}(F)$  and write  $Z = Z_a$ . Take an integer  $d > 0$  large enough that the linear system  $|dH - Z|$  on  $X$  ( $H \subset X$  is a hyperplane section) is very ample. By Bertini's theorem we can take hypersurface sections  $F_0, F_\infty \subset X$  of degree  $d$  satisfying the following conditions:

- (#1)  $F_0 = Z + G$ , where  $G \in \text{Div}(X, C)^+$  such that  $G \cap C$  and  $G \cap Z \cap C = G \cap F \cap C = \emptyset$ .
- (#2)  $F_\infty \cap C$  and  $F_\infty \cap F \cap C = \emptyset$ .

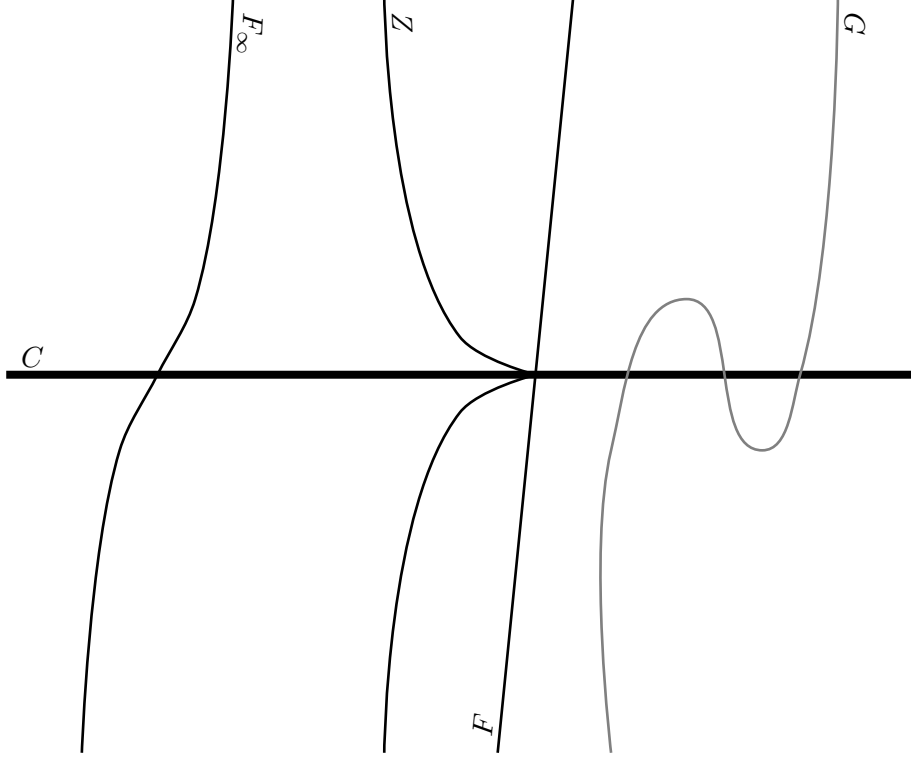


FIGURE 3. The configuration of curves in the proof of Lemma 10.2.

Let  $b$  be a rational function on  $X$  such that

$$(10.2) \quad \operatorname{div}_X(b) = F_0 - F_\infty = Z + G - F_\infty.$$

By (6.2) and (#1) and (#2), locally at  $F \cap C$ , we can write

$$(10.3) \quad b = v(f^e + \pi_D \cdot u) \quad \text{with } u, v \in \mathcal{O}_{X, F \cap C}^\times,$$

**Claim 10.3.** *The natural map*

$$H^0(X, \mathcal{O}_X(-2D + \epsilon F)) \rightarrow \frac{\mathcal{O}_{Z, F \cap C}(-2D + \epsilon F)}{\mathcal{O}_{Z, F \cap C}(-2D + F) + \mathcal{O}_{Z, F \cap C}(-2D - C + \epsilon F)}.$$

*is surjective.*

*Proof.* The target of the above map is isomorphic to

$$\frac{\mathcal{O}_{X, F \cap C}(-2D + \epsilon F)}{\mathcal{O}_{X, F \cap C}(-2D + F) + (I_Z + I_C)\mathcal{O}_{X, F \cap C}(-2D + \epsilon F)}$$

where  $I_W = \mathcal{O}_X(-W)$  for  $W \in \operatorname{Div}(X)^+$ . We consider the following commutative diagram

$$\begin{array}{ccc} H^0(C, \mathcal{O}_C(-2D + \epsilon F)) & \xrightarrow{h_1} & \frac{\mathcal{O}_{C, F \cap C}(-2D + \epsilon F)}{\mathcal{O}_{C, F \cap C}(-2D + F)} \\ \uparrow h_2 & & \simeq \uparrow h_3 \\ H^0(X, \mathcal{O}_X(-2D + \epsilon F)) & \longrightarrow & \frac{\mathcal{O}_{X, F \cap C}(-2D + \epsilon F)}{\mathcal{O}_{X, F \cap C}(-2D + F) + (I_Z + I_C)\mathcal{O}_{X, F \cap C}(-2D + \epsilon F)} \end{array}$$

The map  $h_1$  is surjective since  $H^1(C, \mathcal{O}_C(-2D + F)) = 0$  by the assumption (10.1). The map  $h_2$  is surjective by the assumption (10.1). The map  $h_3$  is an isomorphism since  $\mathcal{O}_{C, F \cap C}(-2D - Z + \epsilon F) \subset \mathcal{O}_{C, F \cap C}(-2D)$ . In fact we have  $\mathcal{O}_{C, F \cap C}(-Z) = \mathcal{O}_{C, F \cap C}(-\epsilon F) \subset \mathcal{O}_{C, F \cap C}(-\epsilon F)$  in view of (6.2) and the assumption  $\epsilon \leq e - 1$ . This proves Claim 10.3.  $\square$

Take  $1 + \frac{\pi_D^2}{f^\epsilon} \alpha \in 1 + \frac{\pi_D^2}{f^\epsilon} \mathcal{O}_{Z, F \cap C}$  with  $\alpha \in \mathcal{O}_{Z, F \cap C}$ . By Claim (10.3) there exist  $a \in H^0(X, \mathcal{O}_X(-2D + \epsilon F))$  and  $\beta, \gamma \in \mathcal{O}_{Z, F \cap C}$  such that

$$a|_Z = \frac{\pi_D^2}{f^\epsilon} \alpha + \frac{\pi_D^2}{f} \beta + \frac{\pi_D^2 \pi}{f^\epsilon} \gamma \in \mathcal{O}_{Z, F \cap C}$$

which implies

$$(10.4) \quad (1+a)|_Z = (1 + \frac{\pi_D^2}{f^\epsilon} \alpha)(1 + \frac{\pi_D^2}{f} \beta')(1 + \frac{\pi_D^2 \pi}{f^\epsilon} \gamma') \in \mathcal{O}_{Z, F \cap C}^\times,$$

where  $\beta' = \beta v^{-1}$  with

$$v = 1 + \frac{\pi_D^2}{f^\epsilon} \alpha = 1 - \pi_D f^{e-\epsilon} u^{-1} \alpha \in \mathcal{O}_{Z, F \cap C}^\times \quad (\text{cf. (6.2)})$$

and  $\gamma' = \gamma v^{-1}(1 + \frac{\pi_D^2}{f} \beta')^{-1} \in \mathcal{O}_{Z, F \cap C}$ . Put

$$Z' = \text{div}_X(1+a) + e'F \quad \text{with } e' = -\text{ord}_F(a) \leq \epsilon.$$

By definition  $Z' \in \text{Div}(X, C)^+$  and  $Z' \cap C \subset F \cap C$  and that locally at  $F \cap C$ ,

$$(10.5) \quad Z' = \text{div}_X(f^{e'} + \pi_D^2 \cdot c) \quad \text{with } c \in \mathcal{O}_{X, F \cap C}.$$

Noting (10.2), Lemma 1.14 implies

$$W(X, C) \ni 0 = \partial\{1+a, b\} = \{1+a\}_Z + \{1+a\}_G - \{1+a\}_{F_\infty} - e'\{b\}_F + \{b\}_{Z'}.$$

By (#1) and (#2),  $a|_G \in \mathcal{O}_{G, G \cap C}(-2D)$  and  $a|_{F_\infty} \in \mathcal{O}_{F_\infty, F_\infty \cap C}(-2D)$  so that

$$\{1+a\}_G, \{1+a\}_{F_\infty} \in F^{(2D)}W(X, C).$$

Hence (10.4) implies

$$\{1 + \frac{\pi_D^2}{f^\epsilon} \alpha\}_Z + \{1 + \frac{\pi_D^2}{f} \beta'\}_Z + \{1 + \frac{\pi_D^2 \pi}{f^\epsilon} \gamma'\}_Z - e'\{b\}_F + \{b\}_{Z'} \in F^{(2D)}W(X, C).$$

The proof of Lemma 10.2 is reduced to showing

$$(10.6) \quad -e'\{b\}_F + \{b\}_{Z'} \in F^{(2D)}W(X, C).$$

**Claim 10.4.**  $b|_{F \cup Z'} \in H^0(F \cup Z', \mathcal{O}_{F \cup Z'}(-D + F_\infty))$ .

*Proof.* Noting  $\text{div}_X(b) = Z + G - F_\infty$  and  $Z' \cap C \subset F \cap C$ , it suffices to show  $b|_{F \cup Z'} \in \mathcal{O}_{F \cup Z', F \cap C}(-D)$ . Noting (10.3) and (10.5), we have locally at  $F \cap C$

$$b = v(f^e + \pi_D u) = v \pi_D (1 - \pi_D f^{e-e'} c u^{-1}) + v f^{e-e'} (f^{e'} + \pi_D^2 c).$$

Since  $e' \leq \epsilon \leq e-1$ , the second term is divisible by a local equation of  $F \cup Z'$ , which proves the claim.  $\square$

Since  $F_\infty$  is ample, we can find  $b' \in H^0(X, \mathcal{O}_X(-D + N F_\infty))$  for  $N$  sufficiently large  $N$  such that  $b'|_{F \cup Z'} = b$ .

**Claim 10.5.**  $\text{div}_X(b') = D - N' F_\infty + W$ , where  $N' \geq 0$  is an integer and  $W$  is an effective Weil divisor. Moreover we have  $W \cap F \cap C = \emptyset$ .

*Proof.* The first assertion is obvious. To show the second assertion, first note  $F \not\subset W$  since  $b'|_F = b|_F \neq 0 \in k(F)$ . Take any  $x \in F \cap C_\lambda$  for  $\lambda \in I$ . Noting  $F_\infty \cap F \cap C = \emptyset$  and  $F \not\subset C$ , the first assertion implies

$$\text{ord}_x(b'|_F) = m_\lambda(F, C_\lambda)_x + (W, F)_x = m_\lambda + (W, F)_x.$$

On the other hand, (10.3) implies  $\text{ord}_x(b'|_F) = \text{ord}_x(b|_F) = m_\lambda$ . This implies  $(W, F)_x = 0$  and the second assertion follows.  $\square$

By Lemma 1.14 we get

$$\begin{aligned} 0 &= \partial\{1+a, b'\} = -e'\{b'\}_F + \{b'\}_{Z'} + \{1+a\}_W - N'\{1+a\}_{F_\infty} \\ &= -e'\{b\}_F + \{b\}_{Z'} + \{1+a\}_W - N'\{1+a\}_{F_\infty}. \end{aligned}$$

Since  $a \in H^0(X, \mathcal{O}_X(-2D + \epsilon F))$  and  $F \cap C \cap F_\infty = F \cap C \cap W = \emptyset$ , we have

$$a|_{F_\infty} \in \mathcal{O}_{F_\infty, C \cap F_\infty}(-2D), \quad a|_W \in \mathcal{O}_{W, C \cap W}(-2D)$$

and hence  $\{1+a\}_W - N'\{1+a\}_{F_\infty} \in F^{(2D)}W(X, C)$ . This implies (10.6) and completes the proof of Lemma 10.2.  $\square$

Let the assumption be as in Lemma 10.1. By (6.2), we have for  $\alpha \in \mathcal{O}_{Z_a, F \cap C}$

$$1 + f^{e+1}\alpha = 1 + \frac{\pi_D^2}{f^{e-1}}u^2\alpha \in 1 + \frac{\pi_D^2}{f^{e-1}}\mathcal{O}_{Z_a, F \cap C}.$$

Thus Lemma 10.2 implies

$$\{1 + f^{e+1}\alpha\}_{Z_a} \in \{1 + \frac{\pi_D^2\pi}{f^{e-1}}\mathcal{O}_{Z_a, F \cap C}\}_{Z_a} + \{1 + \frac{\pi_D^2}{f}\mathcal{O}_{Z_a, F \cap C}\}_{Z_a} + F^{(2D)}W(X, C)$$

and by (6.2), we have

$$1 + \frac{\pi_D^2\pi}{f^{e-1}}\mathcal{O}_{Z_a, F \cap C} \subset 1 + \pi_D\pi\mathcal{O}_{Z_a, F \cap C} = 1 + \mathcal{O}_{Z_a, F \cap C}(-D - C).$$

Hence Lemma 10.1 follows from the following.

**Lemma 10.6.** *Fix a regular closed point  $x \in C$ . Let  $F \in \text{Div}(X, C)^+$  be such that  $x \in F$  and  $F \not\supset C$  at  $x$ . Take a local parameter  $f$  (resp.  $\pi$ ) of  $F$  (resp.  $C$ ) at  $x$  (so that  $(\pi, f)$  is a system of regular parameters at  $x$ ). Let  $Z \in \text{Div}(X, C)^+$  be such that locally at  $x$ ,*

$$(10.7) \quad Z = \text{div}_X(c \cdot f^e + \pi^m) \quad \text{with } c \in \mathcal{O}_{X, x},$$

where  $m > 0$  is an integer. Consider the object  $(\tilde{X}, \tilde{C})$  of  $\mathcal{B}_X$ , where  $g : \tilde{X} \rightarrow X$  is the blowup at  $x$  and  $\tilde{C} = g^*C$  (cf. Definition 1.10). Let  $Z' \subset \tilde{X}$  be the proper transform of  $Z$ .

(1) *If  $e \geq m$ , we have*

$$\frac{\pi_D^2}{f} \in \mathcal{O}_{Z', Z' \cap \tilde{C}}(-g^*(2D - C)).$$

*In particular we have*

$$\{1 + \frac{\pi_D^2}{f}\mathcal{O}_{Z, x}\}_{Z, x} \in F_B^{(2D-C)}W(X, C).$$

(2) *Assuming  $(p, 2m_\lambda) = 1$  for  $\lambda \in I$  with  $x \in C_\lambda$  and  $e \geq m(p^n - 1)$  for an integer  $n > 0$ , we have*

$$\{1 + \frac{\pi_D^2}{f}\mathcal{O}_{Z, x}\}_{Z, x} \in \hat{F}^{(2D)}W(X, C) + p^n \hat{F}^{(1)}W(U).$$

In this section we prove Lemma 10.6(1). The proof of Lemma 10.6(2) will be given in §12. Note that it finishes the proof of Lemma 10.1(1).

Let  $C'$  be the proper transform of  $C$ . We have  $\tilde{C} = C' + E$  with  $E = g^{-1}(x)$ , the exceptional divisor. Let  $F'$  be the proper transform of  $F$  in  $\tilde{X}$ . We claim  $Z' \cap E \cap F' = \emptyset$ . Indeed  $f/\pi$  (resp.  $\pi$ ) is a local parameter of  $F'$  (resp.  $E$ ) at  $F' \cap E$ . By (10.7) and the assumption  $e \geq m$ ,  $Z'$  is defined locally at  $F' \cap E$  by

$c(f/\pi)^e \pi^{e-m} + 1$ . Noting that  $f$  is a local parameter of  $E$  at any  $y \in E - (E \cap F')$ , we have

$$\frac{\pi_D^2}{f} \in \mathcal{O}_{Z', Z' \cap E}(-2g^*D + E) \subset \mathcal{O}_{Z', Z' \cap E}(-g^*(2D - C))$$

noting  $-2g^*D + E = -g^*(2D - C) - C'$ . Lemma 10.6(1) follows from this.

### 11. PROOF OF KEY LEMMA III

Let the notation be as in §6. The following lemma is a preliminary for the proof of Lemma 6.5 whose proof will be completed in §13. In this section we prove only Lemma 11.1(1), which is necessary for the proof of Lemma 12.1. Lemma 10.6(2) and hence Lemma 10.1(2) will be deduced from Lemma 12.1. Lemma 11.1(2) will be then deduced from Lemma 10.1(2). Recall Definition 1.4.

**Lemma 11.1.** *Take a reduced  $Z \in \text{Div}(X, C)^+$  and  $x \in Z \cap C$  and  $\lambda \in I$  with  $x \in C_\lambda$ , and a dense open subset  $V \subset X$  containing  $I$ .*

(1) *We have*

$$\{1 + \mathcal{O}_{Z,x}(-D)\}_{Z,x} \subset F_{\mathbb{A}V}^{(D)}W(X, C) + F_B^{(2D-C)}W(X, C),$$

(2) *Assume  $(p, 2m_\lambda) = 1$ . For any integer  $n > 0$ , we have*

$$\{1 + \mathcal{O}_{Z,x}(-D)\}_{Z,x} \subset F_{\mathbb{A}V}^{(D)}W(X, C) + \widehat{F}^{(2D)}W(X, C) + p^n \widehat{F}^{(1)}W(U).$$

*Remark 11.2.* (1) Lemma 11.1(1) implies

$$\{1 + \mathcal{O}_{Z,x}(-D)\}_{Z,x} \subset F_{\mathbb{A}V}^{(D)}W(X, C) + F_B^{(D+C_0)}W(X, C),$$

where (cf. (6.1))

$$(11.1) \quad C_0 = \sum_{\lambda \in J} C_\lambda \quad \text{with } J = \{\lambda \in I \mid m_\lambda \geq 2\}.$$

(2) Assuming  $p \neq 2$ , Lemma 11.1(1) and (2) imply

$$\{1 + \mathcal{O}_{Z,x}(-D)\}_{Z,x} \subset F_{\mathbb{A}V}^{(D)}W(X, C) + \widehat{F}^{(D+C)}W(X, C) + p^n \widehat{F}^{(1)}W(U).$$

*Proof of Lemma 11.1.* By Bertini's theorem (cf. the argument in the beginning of the proof of Lemma 10.2), there are sections  $F_0, F_\infty \subset X$  in  $\mathcal{L}(d) = |dH|$  for  $d$  sufficiently large, which satisfy the following conditions:

- (#1) the condition (\*) of Lemma 10.1 holds for any  $F \in \mathcal{L}(d)$  (cf. Remark 6.4),
- (#2)  $F_0 = Z + G$  where  $G \in \text{Div}(X, C)^+$  integral,  $G \mathbb{A} C$ ,  $G \cap C \subset V$  and  $G \cap C \cap Z = \emptyset$ ,
- (#3)  $F_\infty \mathbb{A} C$  and  $F_\infty \cap C \subset V$ , and  $F_\infty \cap C \cap F_0 = \emptyset$ .

Let  $L \in \text{Gr}(1, \mathcal{L}(d))$  be the line passing through  $F_0$  and  $F_\infty$  and consider the pencil  $\{F_t\}_{t \in L}$ . We take  $F_\infty$  general so that there exists a finite subset  $\Sigma \subset L$  such that

- (#4)  $F_t \cap F_{t'} \cap C = \emptyset$  for  $t \neq t' \in L$ , and  $F_t \mathbb{A} C$  and  $F_t \cap C \subset V$  for  $t \in L - \Sigma$ .

We take an identification  $L \simeq \mathbb{P}^1 = \text{Proj}(k[T_0, T_1])$  such that  $F_0 = F_t$  with  $t = (1 : 0) \in \mathbb{P}^1$  and  $F_\infty = F_t$  with  $t = (0 : 1) \in \mathbb{P}^1$ . Put  $q = p^N$  for a sufficiently large  $N \in \mathbb{Z}_{>0}$  and take a finite subset  $S \subset \mathbb{P}^1(\mathbb{F}_q) \setminus (\{0, 1, \infty\} \cup \Sigma)$  with  $s = \deg_L(S)$  large enough that the following conditions hold for  $W := Z + F_\infty \in \text{Div}(X, C)^+$ :

- (♠1)  $H^1(W, \mathcal{O}_W(-2D + (s-1)F_\infty)) = 0$ ,
- (♠2)  $H^1(X, \mathcal{O}_X(-D - W + (s-1)F_\infty)) = 0$ .

Take rational functions

$$\rho = \frac{T_0}{T_1 - T_0} \quad \text{and} \quad b = 1 - \rho^{q-1} \quad \text{on } L = \mathbb{P}^1.$$

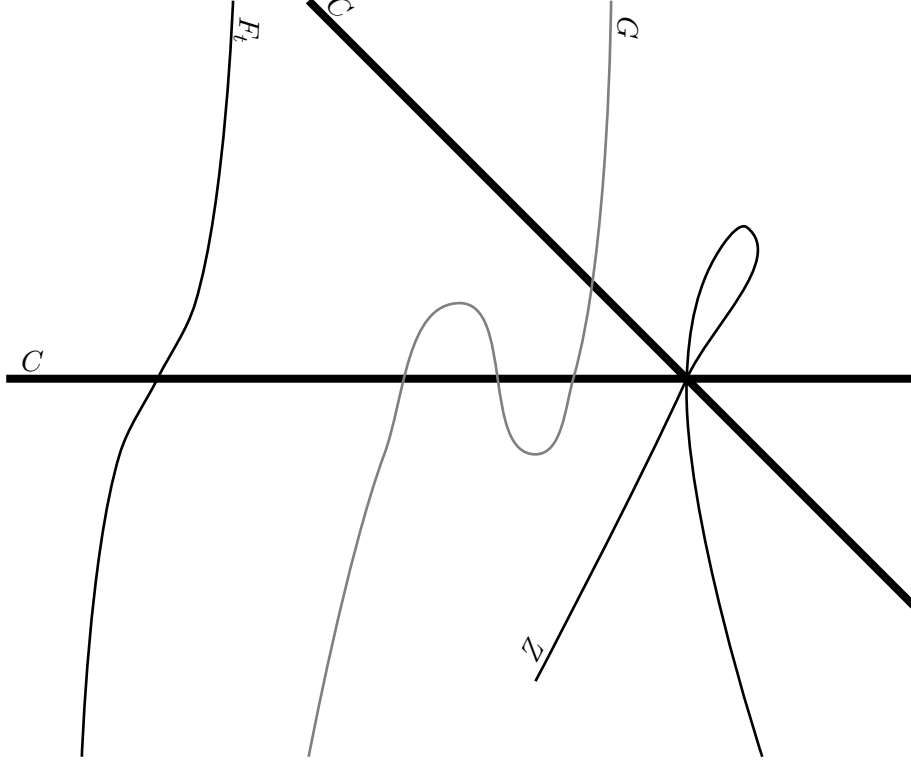


FIGURE 4. The configuration of curves in the proof of Lemma 11.1.

**Claim 11.3.** Put  $\Lambda = \mathbb{P}^1(\mathbb{F}_q) - (S \cup \{0, \infty\})$  and

$$\Theta = -\sum_{t \in \Lambda} F_t + (q-2)F_\infty$$

which is a Cartier divisor on  $X$ . Then the natural map

$$H^0(X, \mathcal{O}_X(-D - \sum_{t \in \Lambda} F_t + (q-2)F_\infty)) \rightarrow \frac{\mathcal{O}_{W, W \cap C}^h(-D + \Theta)}{\mathcal{O}_{W, W \cap C}^h(-2D + \Theta)} = \frac{\mathcal{O}_{Z, Z \cap C}^h(-D)}{\mathcal{O}_{Z, Z \cap C}^h(-2D)} \oplus \frac{\mathcal{O}_{F_\infty, F_\infty \cap C}^h(-D + (q-2)F_\infty)}{\mathcal{O}_{F_\infty, F_\infty \cap C}^h(-2D + (q-2)F_\infty)}$$

is surjective (note that the assumption implies

$$Z \cap C \cap |\Theta| = \emptyset \quad \text{and} \quad \mathcal{O}_{F_\infty, F_\infty \cap C}^h(\Theta) = \mathcal{O}_{F_\infty, F_\infty \cap C}^h((q-2)F_\infty).$$

*Proof.* The above map factors as

$$H^0(X, \mathcal{O}_X(-D + \Theta)) \rightarrow H^0(W, \mathcal{O}_W(-D + \Theta)) \rightarrow \frac{\mathcal{O}_{W, W \cap C}^h(-D + \Theta)}{\mathcal{O}_{W, W \cap C}^h(-2D + \Theta)}.$$

Noting  $\mathcal{O}_X(\Theta) \simeq \mathcal{O}_X((s-1)F_\infty)$ , the first (resp. the second) map is surjective by  $(\spadesuit 2)$  (resp.  $(\spadesuit 1)$ ). This proves Claim 11.3.  $\square$

Since  $F_\infty \cap C$  we have an isomorphism

$$\nu : \frac{\mathcal{O}_{F_\infty, F_\infty \cap C}^h(-D + (q-2)F_\infty)}{\mathcal{O}_{F_\infty, F_\infty \cap C}^h(-D - C + (q-2)F_\infty)} \simeq \bigoplus_{x \in F_\infty \cap C} k(x).$$

Consider the composite map

$$\mu : H^0(X, \mathcal{O}_X(-D + \Theta)) \rightarrow \mathcal{O}_{F_\infty, F_\infty \cap C}^h(-D + (q-2)F_\infty) \xrightarrow{\nu} \bigoplus_{x \in F_\infty \cap C} k(x).$$

By Claim 11.3, for given  $\alpha \in \mathcal{O}_{Z,x}^h(-D)$ , one can find

$$a \in H^0(X, \mathcal{O}_X(-D - \sum_{t \in \Lambda} F_t + (q-2)F_\infty))$$

satisfying the following conditions:

- (♣1)  $a|_Z = \alpha \bmod \mathcal{O}_{Z,x}^h(-2D)$  and  $a|_Z \in \mathcal{O}_{Z,y}^h(-2D)$  for all  $y \in (Z \cap C) - x$ ,
- (♣2)  $\mu(a) = (1, \dots, 1)$ .
- (♣2) implies  $a \in \mathcal{P}_{D,q-2}(F_\infty)$  and we put (cf. Definition 6.2)

$$Z_a = \operatorname{div}_X(1+a) + (q-2)F_\infty.$$

Noting  $b|_{F_\infty} = 1$  and

$$\begin{aligned} (1+a)|_{F_t} &= 1 \quad \text{for } t \in \Lambda = \mathbb{P}^1(\mathbb{F}_q) - (S \cup \{0, \infty\}), \\ \operatorname{div}_X(b) &= \sum_{t \in \mathbb{A}^1(\mathbb{F}_q) - \{0,1\}} F_t + (1-q)F_1 + Z + G, \end{aligned}$$

we get by Lemma 1.14

$$W(X, C) \ni 0 = \partial\{1+a, b\} = \{1+a\}_Z + \{1+a\}_G + \sum_{t \in S} \{1+a\}_{F_t} + \{1-\rho^{q-1}\}_{Z_a}.$$

(♣1) implies

$$\begin{aligned} \{1+a\}_{Z,x} - \{1+\alpha\}_{Z,x} &\in F^{(2D)}W(X, C), \\ \{1+a\}_{Z,y} &\in F^{(2D)}W(X, C) \quad \text{for } y \in (Z \cap C) - x. \end{aligned}$$

By (♣3) and (♣4),

$$a \in \mathcal{O}_{X, C \cap G}(-D) \quad \text{and} \quad a \in \mathcal{O}_{X, C \cap F_t}(-D) \quad \text{for } t \in S.$$

Finally Lemma 10.1(1) implies  $\{1-\rho^{q-1}\}_{Z_a} \in F_B^{(2D-C)}W(X, C)$ . Recalling  $G \cap C$  and  $F_t \cap C$  for  $t \in S$ , this proves Lemma 11.1(1). Lemma 11.1(2) would also follow from Lemma 10.1(2) whose proof will be complete in the next section.  $\square$

## 12. PROOF OF KEY LEMMA IV

In this section we finish the proof of Lemma 10.6(2). Note that this completes the proof of Lemma 10.1 and Lemma 11.1. Lemma 10.6(2) follows from the following lemma where we take  $2D$  in Lemma 10.6 for  $D$  in Lemma 12.1. Let the notation be as in §6.

**Lemma 12.1.** *Fix a regular closed point  $x \in C$  and an integer  $n > 0$ . Let  $F \in \operatorname{Div}(X, C)^+$  be such that  $x \in F$  and  $F \cap C$  at  $x$ . Take a local parameter  $f$  (resp.  $\pi$ ) of  $F$  (resp.  $C$ ) at  $x$  (so that  $(\pi, f)$  is a system of regular parameters at  $x$ ). Let  $Z \in \operatorname{Div}(X, C)^+$  be such that locally at  $x$ ,*

$$(12.1) \quad Z = \operatorname{div}_X(\pi^a + uf^b) \quad \text{with } u \in \mathcal{O}_{X,x}, \quad a, b \in \mathbb{Z}_{>0}.$$

*Assume*

$$(*) \quad (p, m_\lambda) = 1 \text{ for } \lambda \in I \text{ such that } x \in C_\lambda, \text{ and } b \geq a(p^n - 1).$$

*Then we have (cf. Definition 1.4)*

$$\{1 + \mathcal{O}_{Z,x}(-D + F)\}_{Z,x} \in \widehat{F}^{(D)}W(X, C) + p^n \widehat{F}^{(1)}W(U).$$

**Remark 12.2.** The assumption of Lemma 12.1 is satisfied if  $Z$  is regular at  $x$  and  $(Z, C)_x \geq p^n - 1$ .

For the proof of Lemma 12.1, we need the following.

**Lemma 12.3.** *For a fixed integer  $n > 0$ , put*

$$C_P = \sum_{\lambda \in J_P} C_\lambda \quad \text{with} \quad J_P = \{\lambda \in I \mid \frac{m_\lambda}{p^n} \in \mathbb{Z}\}.$$

*For  $Z \in \text{Div}(X, C)^+$  and  $x \in Z \cap C$ , we have*

$$\{1 + \mathcal{O}_{Z,x}(-D)\}_{Z,x} \subset F_{\mathcal{B}}^{(D+C_P)} W(X, C) + p^n F^{(C)} W(X, C).$$

*Proof.* By Lemma 11.1(1) (see also Remark 11.2(1)), the assertion is a consequence of the following fact: for integral  $F \in \text{Div}(X, C)^+$  such that  $F \cap C$  at  $x \in F \cap C$ , we have

$$1 + \mathcal{O}_{F,x}(-D) \subset (1 + \mathcal{O}_{F,x}(-D - C_P)) \cdot (1 + \mathcal{O}_{F,x}(-C))^{p^n},$$

which follows from an isomorphism

$$\frac{1 + \mathcal{O}_{F,x}(-D)}{1 + \mathcal{O}_{F,x}(-D - C_P)} \simeq \prod_{\lambda \in J_P} \frac{1 + \mathcal{O}_{F,x}(-m_\lambda C_\lambda)}{1 + \mathcal{O}_{F,x}(-(m_\lambda + 1)C_\lambda)}$$

and the assumption  $p^n | m_\lambda$  for  $m_\lambda \in J_P$  noting  $\kappa(x)$  is perfect.  $\square$

*Proof of Lemma 12.1* Taking a local parameter  $\pi_D$  of  $D$  at  $x$ , we want to show

$$(12.2) \quad \{1 + \alpha \frac{\pi_D}{f}\}_{Z,x} \in \widehat{F}^{(D)} W(X, C) + p^n \widehat{F}^{(1)} W(U) \quad \text{for } \alpha \in \mathcal{O}_{X,x}.$$

For integers  $m \geq 0$ , we inductively define

$$(X_m, C'_m, E_m, x_m)$$

as follows. For  $m = 0$ ,  $X_0 = X$ ,  $C'_0 = C$ ,  $E_0 = \emptyset$ ,  $x_0 = x$ . For  $m = 1$ , let  $g_1 : X_1 = \text{Bl}_x(X) \rightarrow X$  be the blowup at  $x$ ,  $C'_1 \subset X_1$  be the proper transform of  $C$ ,  $E_1 = g_1^{-1}(x)$  be the exceptional divisor, and  $x_1 = C'_1 \cap E_1$ . Assuming  $(X_{m-1}, C'_{m-1}, E_{m-1}, x_{m-1})$  defined, let  $g_m : X_m = \text{Bl}_{x_{m-1}}(X_{m-1}) \rightarrow X_{m-1}$ ,  $C'_m \subset X_m$  be the proper transform of  $C'_{m-1}$ ,  $E_m = g_m^{-1}(x_{m-1})$ , and  $x_m = C'_m \cap E_m$ . Let

$$\phi_m = g_m \circ \cdots \circ g_1 : X_m \rightarrow X,$$

be the composite map,  $C_m = \phi_m^{-1}(C)_{\text{red}}$ , and  $E_{i,m} \subset X_m$  be the proper transform of  $E_i \subset X_i$  for  $1 \leq i \leq m-1$ . We also define  $E_{0,1}$  as the proper transform of  $F$ . Note that  $(X_m, C_m)$  is an object of  $\widehat{\mathcal{B}}_X$  but not in  $\mathcal{B}_X$  (cf. Definition 1.10). We easily see that the following facts hold for  $m \geq 1$ :

- (\*1)  $(\pi/f^m, f)$  is a system of regular parameters of  $X_m$  at  $x_m$ .
- (\*2)  $f$  is a local parameter of  $E_m \subset X_m$  at any point  $y \in E_m \setminus E_{m-1,m}$ .
- (\*3) Let  $Z_m \subset X_m$  be the proper transform of  $Z$  in (12.1). If  $b \geq am$ ,  $Z_m$  is defined locally around  $E_m \setminus E_{m-1,m}$  by  $(\pi/f^m)^a + u f^{b-am}$ . We have

$$b \geq am \iff Z_m \cap \phi_m^{-1}(x) \subset E_m \setminus E_{m-1,m},$$

$$b \geq am + 1 \iff Z_m \cap \phi_m^{-1}(x) = x_m.$$

$$(*4) \quad \phi_m^* C = C'_m + m E_m + \sum_{1 \leq i \leq m-1} i E_{i,m},$$

By the assumption  $(p, m_\lambda) = 1$ , one can take an integer  $m$  such that  $1 \leq m \leq p^n - 1$  and  $p^n | mm_\lambda - 1$ . Take  $y \in Z_m \cap \phi_m^{-1}(x)$ . By the assumption (\*) of Lemma 12.1, we have  $b \geq a(p^n - 1) \geq am$  and (\*2), (\*3) and (\*4) imply

$$\frac{\pi_D}{f} \in \mathcal{O}_{Z_m,y}(-\phi_m^* D + E_m) \quad \text{and} \quad \phi_m^* D - E_m \geq 0$$

and hence

$$\{1 + \alpha \frac{\pi_D}{f}\}_{Z_m,y} \in F^{(\phi_m^* D - E_m)} W(X_m, C_m) \quad \text{for } \alpha \in \mathcal{O}_{X,x}.$$

By (\*4) the multiplicity of  $E_m$  in  $\phi_m^* D - E_m$  is  $mm_\lambda - 1$ . Since  $p^n | mm_\lambda - 1$ , Lemma 12.3 with Remark 1.13 implies

$$\{1 + \alpha \frac{\pi_D}{f}\}_{Z_m, y} \in F_B^{(\phi_m^* D)} W(X_m, C_m) + p^n F^{(C_m)} W(X_m, C_m).$$

Noting  $F^{(C_m)} W(X_m, C_m) \subset \widehat{F}^{(1)} W(U)$ , this implies (12.2) and completes the proof of Lemma 12.1.

### 13. PROOF OF KEY LEMMA V

In this section we prove Lemma 6.5. Indeed we prove the following.

**Lemma 13.1.** *For any positive integers  $n, N > 0$ , we have*

$$\widehat{F}^{(D)} W(X, C) \subset F_{\mathbb{M}V}^{(D)} W(X, C) + \widehat{F}^{(D+N \cdot C)} W(X, C) + p^n \widehat{F}^{(1)} W(U).$$

For the proof we need two preliminary lemmas 13.2 and 13.5. Put (cf. (6.1))

$$(13.1) \quad C_0 = \sum_{\lambda \in J} C_\lambda \quad \text{with } J = \{\lambda \in I \mid m_\lambda \geq 2\}.$$

**Lemma 13.2.** *For any integer  $N > 0$ , we have (cf. Definition 1.4)*

$$F_B^{(D)} W(X, C) \subset F_{\mathbb{M}V}^{(D)} W(X, C) + F_B^{(D+N \cdot C_0)} W(X, C).$$

*Proof.* By induction on  $N$ , we may assume  $N = 1$ . Lemma 11.1(1) implies

$$(13.2) \quad F^{(D)} W(X, C) \subset F_{\mathbb{M}V}^{(D)} W(X, C) + F_B^{(D+C_0)} W(X, C).$$

Hence it suffices to show

$$F_B^{(D)} W(X, C) \subset F^{(D)} W(X, C) + F_B^{(D+C_0)} W(X, C),$$

which follows from the following.

**Claim 13.3.** *Let  $g : \tilde{X} \rightarrow X$  be a successive blowups in  $\mathcal{B}_X$  (cf. Definition 1.9(2)). Note that  $\tilde{C} = g^*(C)$  is reduced. Then*

$$F^{(g^* D)} W(\tilde{X}, \tilde{C}) \subset F^{(D)} W(X, C) + F_B^{(D+C_0)} W(X, C).$$

First we assume  $g$  is a blowup at a regular closed point  $x$  of  $C$ . Lemma 11.1(1) applied to  $(\tilde{X}, \tilde{C})$  implies

$$F^{(g^* D)} W(\tilde{X}, \tilde{C}) \subset F_{\mathbb{M}}^{(g^* D)} W(\tilde{X}, \tilde{C}) + F_B^{(g^* D + \tilde{C}_0)} W(\tilde{X}, \tilde{C}),$$

where  $\tilde{C}_0$  is defined for  $g^* D$  as (13.1). Clearly  $\tilde{C}_0 = g^* C_0$  so that

$$F_B^{(g^* D + \tilde{C}_0)} W(\tilde{X}, \tilde{C}) = F_B^{(g^*(D+C_0))} W(\tilde{X}, \tilde{C}) = F_B^{(D+C_0)} W(X, C),$$

where the second equality holds by (1.3). Hence it suffices to show the following.

**Claim 13.4.** *We have*

$$F_{\mathbb{M}}^{(g^* D)} W(\tilde{X}, \tilde{C}) \subset F^{(D)} W(X, C) + F_B^{(D+C)} W(X, C).$$

It suffices to show

$$\{1 + \mathcal{O}_{F,y}^h(-g^* D)\}_{F,y} \subset F^{(D)} W(X, C) + F_B^{(D+C)} W(X, C)$$

for an integral  $F \in \text{Div}(\tilde{X}, \tilde{C})^+$  and  $y \in F \cap \tilde{C}$  such that  $F \pitchfork \tilde{C}$  at  $y$ . Let  $x = g(y)$ . First we show that we may assume  $\kappa(x) = \kappa(y)$ . Take a finite Galois extension  $k'$  of  $k$  and consider the diagram

$$\begin{array}{ccc} \tilde{X}' & \xrightarrow{\tilde{\phi}} & \tilde{X} \\ g' \downarrow & & \downarrow g \\ X' & \xrightarrow{\phi} & X \end{array}$$

where  $X' = X \otimes_k k'$  and  $\tilde{X}' = \tilde{X} \otimes_k k'$ . Take a point  $y' \in \tilde{X}'$  lying over  $y$  and let  $x' = g'(y')$ . Taking  $k'$  large enough, we may assume  $\kappa(x') = \kappa(y')$ . Take  $F' \in \text{Div}(\tilde{X}', \tilde{C}')^+$  which is integral and finite étale over  $F$ . The norm maps (1.2) for  $\tilde{\phi}$  and  $\phi$  induce a commutative diagram

$$\begin{array}{ccc} 1 + \mathcal{O}_{F', y'}^h(-\tilde{\phi}^* \tilde{D}) & \xrightarrow{N_{F'/F}} & 1 + \mathcal{O}_{F, y}^h(-\tilde{D}) \\ \downarrow \{ , \}_{F', y'} & & \downarrow \{ , \}_{F, y} \\ F(\tilde{\phi}^* \tilde{D})W(\tilde{X}', \tilde{C}') & \xrightarrow{N_{\tilde{\phi}}} & F(\tilde{D})W(\tilde{X}, \tilde{C}) \\ \uparrow \hookrightarrow & & \uparrow \hookrightarrow \\ F(\phi^* D)W(X', C') & \xrightarrow{N_{\phi}} & F(D)W(X, C) \end{array}$$

where  $C'$  (resp.  $\tilde{C}'$ ) is the reduced preimage of  $C$  in  $X'$  (resp.  $\tilde{X}'$ ) and  $\tilde{D} = g^*D$ , and  $N_{F'/F}$  is induced by the norm map  $k(F')^\times \rightarrow k(F)^\times$ . Since  $F' \rightarrow F$  is étale,  $N_{F'/F}$  is surjective. Thus Claim 13.4 is reduced to showing

$$\{1 + \mathcal{O}_{F', y'}^h(-g'^* D')\}_{F', y'} \subset F^{(D')}W(X', C') + F_B^{(D'+C')}W(X', C')$$

where  $D' = \phi^*D$ . Hence we may prove Claim 13.4 assuming  $\kappa(x) = \kappa(y)$ . By Lemma 13.10 below there exists  $G \in \text{Div}(X, C)^+$  such that  $G$  is regular at  $x$  and  $(G', F)_y \geq 2$  where  $G'$  is the proper transform of  $G$  in  $\tilde{X}$ . Noting  $\tilde{C} = g^*C$ , Lemma 9.1 applied to  $(\tilde{X}, \tilde{C})$  and (1.3) imply

$$\{1 + \mathcal{O}_{F, y}^h(-g^*D)\}_{F, y} \subset \{1 + \mathcal{O}_{G', y}^h(-g^*D)\}_{G', y} + F_B^{(D+C)}W(X, C).$$

Since  $G$  is regular at  $x = g(y)$ ,  $G' \rightarrow G$  is an isomorphism at  $y$ . Hence we have

$$\{1 + \mathcal{O}_{G', y}^h(-g^*D)\}_{G', y} = \{1 + \mathcal{O}_{G, x}^h(-D)\}_{G, x} \text{ in } W(U).$$

This completes the proof of Claim 13.4.

Now we prove Claim 13.3 in general case by induction on the number of blown-up points. Decompose  $g$  as

$$g : \tilde{X} \xrightarrow{\psi} X' \xrightarrow{\phi} X \quad \text{with } D' = \phi^*D, \tilde{D} = \psi^*D' = g^*D,$$

where  $\phi$  is in  $\mathcal{B}_X$  and  $\psi$  is a blowup at a regular closed point of  $C'$ . By the induction hypothesis applied for  $\phi$ , we have

$$(13.3) \quad F^{(D')}W(X', C') \subset F^{(D)}W(X, C) + F_B^{(D+C_0)}W(X, C),$$

By applying to  $\psi$  what we have shown, we get

$$F^{(\tilde{D})}W(\tilde{X}, \tilde{C}) \subset F^{(D')}W(X', C') + F_B^{(D'+C'_0)}W(X', C'),$$

where  $C'_0$  is defined for  $D'$  as (13.1). Noting that  $\phi$  is in  $\mathcal{B}_X$ , we see  $\phi^*C_0 = C'_0$  and

$$F_B^{(D'+C'_0)}W(X', C') = F_B^{(\phi^*(D+C_0))}W(X', C') = F_B^{(D+C_0)}W(X, C).$$

By (13.3), this completes the proof of Claim 13.3.  $\square$

**Lemma 13.5.** *For any integer  $N > 0$ , we have*

$$\widehat{F}^{(D)}W(X, C) \subset F_{\mathbb{A}^1V}^{(D)}W(X, C) + \widehat{F}^{(D+N \cdot C_0)}W(X, C).$$

*Proof.* We fix  $N$ . By (13.2) it suffices to show

$$\widehat{F}^{(D)}W(X, C) \subset F^{(D)}W(X, C) + \widehat{F}^{(D+N \cdot C_0)}W(X, C),$$

which follows from the following.

**Claim 13.6.** *Let  $g : \tilde{X} \rightarrow X$  be a successive blowups at closed points and  $\tilde{C} = g^{-1}(C)_{\text{red}}$  and  $\tilde{D} = g^*D \in \text{Div}(\tilde{X}, \tilde{C})^+$ . Then*

$$F^{(\tilde{D})}W(\tilde{X}, \tilde{C}) \subset F^{(D)}W(X, C) + \widehat{F}^{(D+N \cdot C_0)}W(X, C).$$

First we assume  $g$  is a blowup at a closed point  $x \in X$ . By Lemma 13.2 applied to  $(\tilde{X}, \tilde{C})$  and  $\tilde{D}$ , we have

$$F^{(\tilde{D})}W(\tilde{X}, \tilde{C}) \subset F_{\mathbb{A}^1}^{(\tilde{D})}W(\tilde{X}, \tilde{C}) + F_{\mathcal{B}}^{(\tilde{D}+M \cdot \tilde{C}_0)}W(\tilde{X}, \tilde{C})$$

for any integer  $M > 0$  where  $\tilde{C}_0$  is defined for  $\tilde{D}$  as (13.1). We have  $|g^{-1}(C_0)| \subset |\tilde{C}_0|$  so that  $M \cdot \tilde{C}_0 \geq N \cdot g^*C_0$  for  $M$  large enough. Then

$$F_{\mathcal{B}}^{(\tilde{D}+M \cdot \tilde{C}_0)}W(\tilde{X}, \tilde{C}) \subset F_{\mathcal{B}}^{(g^*(D+N \cdot C_0))}W(\tilde{X}, \tilde{C}) \subset \widehat{F}^{(D+N \cdot C_0)}W(X, C).$$

Thus we are reduced to showing the following.

**Claim 13.7.** *We have*

$$F_{\mathbb{A}^1}^{(\tilde{D})}W(\tilde{X}, \tilde{C}) \subset F^{(D)}W(X, C) + \widehat{F}^{(D+N \cdot C)}W(X, C).$$

It suffices to show

$$\{1 + \mathcal{O}_{F,y}^h(-\tilde{D})\}_{F,y} \subset F^{(D)}W(X, C) + F^{(g^*(D+N \cdot C))}W(\tilde{X}, \tilde{C}).$$

for an integral  $F \in \text{Div}(\tilde{X}, \tilde{C})^+$  and  $y \in F \cap \tilde{C}$  such that  $F \mathbb{A}^1 \tilde{C}$  at  $y$ . We may assume  $x = g(y)$ . By the same argument as the proof of Claim 13.4, we may assume  $\kappa(x) = \kappa(y)$ . By Lemma 13.10 below there exists  $G \in \text{Div}(X, C)^+$  such that  $G$  is regular at  $x$  and  $(G', F)_y \geq m + 1$  where  $G'$  is the proper transform of  $G$  in  $\tilde{X}$  and  $m$  is any positive integer. Lemma 9.1 implies

$$\{1 + \mathcal{O}_{F,y}^h(-\tilde{D})\}_{F,y} \subset \{1 + \mathcal{O}_{G',y}^h(-\tilde{D})\}_{G',y} + F^{(\tilde{D}+m \cdot \tilde{C})}W(\tilde{X}, \tilde{C}).$$

We have  $m \tilde{C} \geq N \cdot g^*C$  for  $m$  sufficiently large and hence we get

$$\{1 + \mathcal{O}_{F,y}^h(-\tilde{D})\}_{F,y} \subset \{1 + \mathcal{O}_{G',y}^h(-\tilde{D})\}_{G',y} + F^{(g^*(D+N \cdot C))}W(\tilde{X}, \tilde{C}).$$

Claim 13.7 follows by the same argument as the last part of the proof of Claim 13.4.

Now we prove Claim 13.6 in general case by induction on the number of blown-up points. Decompose  $g$  as

$$g : \tilde{X} \xrightarrow{\psi} X' \xrightarrow{\phi} X \quad \text{with } D' = \phi^*D, \tilde{D} = \psi^*D' = g^*D,$$

where  $\psi$  is a blowup at a closed point. By the induction hypothesis applied for  $\phi$ , we have

$$(13.4) \quad F^{(D')}W(X', C') \subset F^{(D)}W(X, C) + \widehat{F}^{(D+N \cdot C_0)}W(X, C),$$

By applying to  $\psi$  what we have shown, we get

$$(13.5) \quad F^{(\tilde{D})}W(\tilde{X}, \tilde{C}) \subset F^{(D')}W(X', C') + \widehat{F}^{(D'+M \cdot C'_0)}W(X', C')$$

for any integer  $M > 0$ , where  $C'_0$  is defined for  $D'$  as (13.1). We have  $|\phi^{-1}(C_0)| \subset |C'_0|$  so that  $M \cdot C'_0 \geq N \cdot \phi^*C_0$  for  $M$  large enough. Hence

$$\widehat{F}^{(D'+M \cdot C'_0)}W(X', C') \subset \widehat{F}^{(\phi^*(D+N \cdot C_0))}W(X', C') = \widehat{F}^{(D+N \cdot C_0)}W(X, C).$$

Thus Claim 13.6 follows from (13.4) and (13.5).  $\square$

*Remark 13.8.* Lemma 13.5 implies that if  $D \geq 2C$ , we have

$$\widehat{F}^{(D)}W(X, C) \subset F_{\mathbb{A}^1 V}^{(D)}W(X, C) + \widehat{F}^{(D+N \cdot C)}W(X, C) \quad \text{for any } N > 0.$$

Now we finish the proof of Lemma 13.1. We fix  $N > 0$ . By Remark 11.2(2) we have

$$F^{(D)}W(X, C) \subset F_{\mathbb{A}^1 V}^{(D)}W(X, C) + \widehat{F}^{(D+C)}W(X, C) + p^n \widehat{F}^{(1)}W(U).$$

Noting  $D + C \geq 2C$ , Remark 13.8 implies

$$\widehat{F}^{(D+C)}W(X, C) \subset F_{\mathbb{A}^1 V}^{(D)}W(X, C) + \widehat{F}^{(D+N \cdot C)}W(X, C).$$

Thus it suffices to show

$$\widehat{F}^{(D)}W(X, C) \subset F^{(D)}W(X, C) + \widehat{F}^{(D+N \cdot C)}W(X, C) + p^n \widehat{F}^{(1)}W(U),$$

which follows from the following.

**Claim 13.9.** *For  $g : \tilde{X} \rightarrow X$ ,  $\tilde{C}$  and  $\tilde{D}$  as in Claim 13.6, we have*

$$F^{(\tilde{D})}W(\tilde{X}, \tilde{C}) \subset F^{(D)}W(X, C) + \widehat{F}^{(D+N \cdot C)}W(X, C) + p^n \widehat{F}^{(1)}W(U).$$

*Proof.* By the same induction argument as the last part of the proof of Claim 13.6, we are reduced to the case where  $g$  is a blowup at a closed point  $x \in X$ . By Remark 11.2(2) applied to  $\tilde{X}, \tilde{C}, \tilde{D}$ , we have

$$F^{(\tilde{D})}W(\tilde{X}, \tilde{C}) \subset F_{\mathbb{A}^1}^{(\tilde{D})}W(\tilde{X}, \tilde{C}) + \widehat{F}^{(\tilde{D}+\tilde{C})}W(\tilde{X}, \tilde{C}) + p^n \widehat{F}^{(1)}W(U).$$

Noting  $\tilde{D} + \tilde{C} \geq 2\tilde{C}$ , Remark 13.8 applied to  $\tilde{X}, \tilde{C}, \tilde{D} + \tilde{C}$  implies

$$\widehat{F}^{(\tilde{D}+\tilde{C})}W(\tilde{X}, \tilde{C}) \subset F_{\mathbb{A}^1}^{(\tilde{D})}W(\tilde{X}, \tilde{C}) + \widehat{F}^{(\tilde{D}+M \cdot \tilde{C})}W(\tilde{X}, \tilde{C}) \quad \text{for any } M > 0.$$

Taking  $M$  large enough that  $M \cdot \tilde{C} \geq N \cdot g^*C$ , we have

$$\widehat{F}^{(\tilde{D}+M \cdot \tilde{C})}W(\tilde{X}, \tilde{C}) \subset \widehat{F}^{(g^*(D+N \cdot C))}W(\tilde{X}, \tilde{C}) = \widehat{F}^{(D+N \cdot C)}W(X, C).$$

Thus we get

$$F^{(\tilde{D})}W(\tilde{X}, \tilde{C}) \subset F_{\mathbb{A}^1}^{(\tilde{D})}W(\tilde{X}, \tilde{C}) + \widehat{F}^{(D+N \cdot C)}W(X, C) + p^n \widehat{F}^{(1)}W(U).$$

Now Claim 13.9 follows from Claim 13.7.  $\square$

**Lemma 13.10.** *Let  $X$  be a smooth surface over  $k$  and  $x \in X$  be a closed point and  $g : X' \rightarrow X$  be the blowup at  $x$  with  $E = g^{-1}(x)$ . Let  $x'$  be a closed point of  $E$  such that  $\kappa(x) = \kappa(x')$ . Let  $F \subset X'$  be a curve such that  $x' \in F$  and  $F \cap E$  at  $x'$ . Then, for any integer  $m > 0$ , there exists a curve  $G \subset X$  such that  $G$  is regular at  $x$  and  $(G', F)_{x'} \geq m$  where  $G'$  is the proper transform of  $G$  in  $X'$ .*

*Proof.* By the assumption,  $g(F)$  has a regular analytic branch  $T \subset \text{Spec}(\widehat{\mathcal{O}_{X,x}})$  whose proper transform in  $\hat{X}' := X' \times_X \text{Spec}(\widehat{\mathcal{O}_{X,x}})$  is the restriction of  $F$  to  $\hat{X}'$ . Then it suffices to take  $G$  such that  $G$  is regular at  $x$  and  $(G, T)_x \geq m + 1$ .  $\square$

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